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THE GENERALIZED WEIERSTRASS APPROXIMATION THEOREM

by Marshall H. Stone

(Continued from March-April issue)

5. *Extension to Complex Functions.* It is natural to consider the extension of the preceding results to the case of complex-valued functions. The fact that the complex numbers are not ordered is an obstacle to the introduction of lattice operations for complex-valued functions. Accordingly the results of §2 do not lend themselves to extension in the desired sense, unless they are first expressed in terms of the operation of forming the absolute value. However it is easy to see that the complex-linear operations (addition and multiplication by complex numbers) and the operation of forming the absolute value do not work well enough together for us to obtain any very interesting or useful extension to the complex case. Matters appear quite differently when we consider the linear ring operations. In fact, we find here that extremely interesting new possibilities, quite beyond the scope of the present inquiry, are immediately opened up. For example, the theory of analytic functions can be considered as an answer to the question, "If X is a bounded closed subset of the complex plane, and $\mathfrak{X}_0$ is the family of all polynomials in the complex variable z, what functions can be uniformly approximated on X by functions in $\mathfrak{X}_0$?" As this observation clearly suggests, a full investigation of the complex case would be both difficult and rewarding. In order to limit ourselves to considerations of the kind already met in the preceding sections, we shall include the operation of forming conjugates along with the linear ring operations in our examination of the complex case. Accordingly, if $\mathfrak{X}$ is the family of all bounded continuous complex-valued functions on a topological space X and $\mathfrak{X}_0$ is an arbitrary subfamily of $\mathfrak{X}$, we designate by $\mathfrak{U}(\mathfrak{X}_0)$ the family of all those functions which can be obtained from $\mathfrak{X}_0$ by the linear ring operations, the operation of forming conjugates, and uniform passages to the limit. It is easily seen that $\mathfrak{U}(\mathfrak{X}_0)$ is obtainable by using first the algebraic operations and then a single passage to the limit; and that $\mathfrak{U}(\mathfrak{X}_0)$ is closed under all the operations permitted. If we designate the conjugate by means of a dash, so that $\bar{f}$ is the function whose value $\bar{f}(x)$ at x is equal to the conjugate of the complex number $f(x)$, we can define two related operations, namely those of forming the real part and the imaginary part of f, by the equations

$$\Re f = \tfrac{1}{2}(f + \bar{f}) \qquad \Im f = 1/2i(f - \bar{f})$$

from which the relations

$$f = \Re f + i\,\Im f, \qquad \bar{f} = \Re f - i\,\Im f$$

follow directly. The functions $\Re f$ and $\Im f$ are real-valued continuous functions which belong to $\mathfrak{U}(\mathfrak{X}_0)$ whenever f is in $\mathfrak{X}_0$. It is easy to see that $\mathfrak{U}(\mathfrak{X}_0)$ can be obtained in the following manner: we first form the family $\mathfrak{Y}_0$ of all real-valued functions expressible as $\Re f$ or $\Im f$ where f is in $\mathfrak{X}_0$; we then form the family $\mathfrak{B}(\mathfrak{Y}_0)$ of all those (real) functions which can be obtained from $\mathfrak{Y}_0$ by the real linear ring operations and uniform passage to the limit; and finally we find $\mathfrak{U}(\mathfrak{X}_0)$ as the family of all functions $f + ig$ where f and g are in $\mathfrak{B}(\mathfrak{Y}_0)$. In view of this observation we can carry Theorem 5 and its

corollaries over to the complex case without any further difficulty. The results will be given without further discussion, as follows.

Theorem 10: *Let X be a compact space, $\mathfrak{X}$ the family of all continuous complex functions on X, $\mathfrak{X}_0$ an arbitrary subfamily of $\mathfrak{X}$, and $\mathfrak{U}(\mathfrak{X}_0)$ the family of all functions (necessarily continuous) generated from $\mathfrak{X}_0$ by the linear ring operations, the operation of forming the conjugate, and uniform passage to the limit. Then a necessary and sufficient condition for a function f in $\mathfrak{X}$ to be in $\mathfrak{U}(\mathfrak{X}_0)$ is that f satisfy every linear relation of the form $g(x) = 0$ or $g(x) = g(y)$ which is satisfied by all functions in $\mathfrak{X}_0$. If $\mathfrak{X}_0$ is a closed linear subring of $\mathfrak{X}$ which contains $\bar{f}$ together with f — that is, if $\mathfrak{X}_0 = \mathfrak{U}(\mathfrak{X}_0)$ — then $\mathfrak{X}_0$ is characterized by the system of all the linear relations of this kind which are satisfied by every function belonging to it. In other words, $\mathfrak{X}_0$ is characterized by the partition of X into mutually disjoint closed subsets on each of which every function in $\mathfrak{X}_0$ is constant and by the specification of that one, if any, of these subsets on which every function in $\mathfrak{X}_0$ vanishes.*

Corollary 1: *In order that $\mathfrak{U}(\mathfrak{X}_0)$ contain a non-vanishing constant function it is necessary and sufficient that for every x in X there exist some f in $\mathfrak{X}_0$ such that $f(x) \neq 0$.*

Corollary 2: *If $\mathfrak{X}_0$ is a separating family for X, then $\mathfrak{U}(\mathfrak{X}_0)$ either coincides with $\mathfrak{X}$ or is, for a uniquely determined point x_0, the family of all functions f in $\mathfrak{X}$ such that $f(x_0) = 0$. If, conversely, $\mathfrak{X}$ is a separating family for X and $\mathfrak{U}(\mathfrak{X}_0)$ either coincides with $\mathfrak{X}$ or is the family of all those f in $\mathfrak{X}$ which vanish at some fixed point x_0 in X, then $\mathfrak{X}_0$ is a separating family.*

In the case of complex-valued functions, the definition of an ideal remains the same: a non-void subclass $\mathfrak{X}_0$ of $\mathfrak{X}$ is said to be an ideal if $\mathfrak{X}_0$ contains $f + g$ whenever it contains both f and g, and $\mathfrak{X}_0$ contains fg whenever it contains f, g being an arbitrary function in $\mathfrak{X}$. Clearly, an ideal is closed under the linear ring operations, since multiplication by a complex number is equivalent to multiplication by a constant complex-valued function. Now if an ideal $\mathfrak{X}_0$ is closed under uniform passages to the limit we can show that $\mathfrak{X}_0$ contains $\bar{f}$ together with f. For an arbitrary function f in $\mathfrak{X}_0$ we define a function g_n by putting $g_n(x) = f(x)\,|f(x)|^{1/n}/f(x)$ when $f(x) \neq 0$ and $g_n(x) = 0$ when $f(x) = 0$. If x is any point in X and x_0 any point such that $f(x_0) = 0$ we have

$$|g_n(x)| = |f(x)|^{1/n}, \qquad |g_n(x) - g_n(x_0)| = |f(x) - f(x_0)|^{1/n},$$

whence g_n is bounded on X and continuous at x_0. On the other hand, it is evident that g_n is continuous at any point x where $f(x) \neq 0$. Thus g_n is a function in $\mathfrak{X}$ and fg_n a function in the ideal $\mathfrak{X}_0$. The function $h_n = |f - fg_n| = |\bar{f}(1 - |f|^{1/n})| = ||f| - |f|^{1+1/n}|$ is evidently a real continuous function on X which converges as $n \to \infty$ to the function everywhere equal to zero. for any x it is easily verified that $h_n(x)$ is a non-increasing sequence. Accordingly Corollary 1 to Theorem 1 shows that the convergence is uniform. It follows that fg_n converges uniformly to $\bar{f}$ and hence that $\bar{f}$ is in the closed ideal $\mathfrak{X}_0$. As a result we can state without further preliminaries the following extension of Theorem 8 from the real to the complex case, essentially due to Šilov.

Theorem 11: (Šilov [3]). *Let $\mathfrak{X}$ be the linear ring of all the continuous*

complex functions on a compact space X, let $\mathfrak{X}_0$ be an arbitrary non-void subfamily of $\mathfrak{X}$, let X_0 be the closed set of all those points x at which every function f in $\mathfrak{X}_0$ vanishes, and let $\mathfrak{Y}_0$ be the family of all those functions f in $\mathfrak{X}$ which vanish at every point of X_0. Then $\mathfrak{Y}_0$ is the smallest closed ideal containing $\mathfrak{X}_0$; and $\mathfrak{X}_0$ is a closed ideal if and only if $\mathfrak{X}_0 = \mathfrak{Y}_0$. A closed ideal $\mathfrak{X}_0$ is characterized by the associated closed set X_0; in particular, $\mathfrak{X}_0 = \mathfrak{Y}_0 = \mathfrak{X}$ if and only if X_0 is void.

Proof: It is obvious that $\mathfrak{Y}_0$ is an ideal. If a closed ideal $\mathfrak{X}_1$ contains $\mathfrak{X}_0$ then by virtue of what we have just proved above it is a linear subring which contains $\bar{f}$ along with f; and hence a simple argument based on Theorem 10 shows that $\mathfrak{X}_1$ contains $\mathfrak{Y}_0$. The remainder of the theorem follows then in a familiar way.

6. *The Extension to Locally Compact Spaces.* A natural question arises as to the possibility of relaxing the topological conditions imposed hitherto upon the space X. A thorough examination of this question would take us too far afield. Suffice it to say that a great deal of light can be thrown on this question by applying the theory of compactification developed by the writer in [1] and by Čech in a later paper [4]. In fact, it can be said that this theory allows us to solve the problem of approximation in the family $\mathfrak{X}$ of all bounded continuous real (or complex) functions on an *arbitrary* topological space X — in exactly the same sense that the problem has been solved above for compact X. The essence of the method alluded to in these remarks is to replace X by a suitable compactification X^*, extending every function in $\mathfrak{X}$ over the compact space X^* without sacrifice of its continuity. There is however, one very special instance of sufficient immediate interest for us to pay it some attention here. This is the case where X is a locally compact space.

By definition a space is locally compact if every point of the space is interior to some compact subset of the space. Typical examples of spaces which are locally compact without being compact are afforded by the euclidean spaces of n dimensions. If X is a locally compact space which is not compact it can be compactified, as is well known, by the adjunction of a single point. Specifically, we adjoin an element x_∞ to X, obtaining the set X^*, and we define a subset U^* of X^* to be open if it is a subset of X and is open in X, or if it is the complement of a *closed* compact subset of X (note that, while a closed subset of a compact space is compact, a compact subset of a topological space is not necessarily closed unless it is a Hausdorff space!). The totality of open sets in X^* is easily verified to have the properties normally required: X^* and its void subset are open; the union of any family of open sets is open; and the intersection of any finite family of open sets is open. Moreover, X^* can be shown to be compact, as follows: in any family of open sets whose union is X^* we can find one open set containing x_∞; its complement X_0 is compact and is contained in the union of the remaining open sets in the family; but then a finite number of the latter must, because X_0 is compact, have a union containing X_0.

In order that a real function f defined on X should agree there with a function f^* defined and continuous on X^*, it is necessary and sufficient that the function f, in addition to being continuous on X, should satisfy the inequality $|f(x) - f(y)| < \epsilon$ for all x and y outside a suitable closed compact

set X_ϵ. The necessity of the condition is obvious: when f^* exists, there is an open set U_ϵ^* containing x_∞ such that for all x and y in it $|f^*(x) - f^*(x_\infty)| < \epsilon/2$, $|f^*(y) - f^*(x_\infty)| < \epsilon/2$, and hence $|f(x) - f(y)| = |f^*(x) - f^*(y)| < \epsilon$; the appropriate set X_ϵ is therefore the complement of U_ϵ^*. The sufficiency of the condition is easy to prove aside from the determination of the value which should be assigned to f^* at x_∞. Let Z be the closure of the set of all real numbers $\zeta = f(x)$, where x is in X and is restricted to lie in a fixed open subset of X^* containing x_∞. Since X is not compact the intersection of any finite number of such open sets must contain points of X, and the corresponding sets Z therefore have a common point. Thus any finite number of the sets Z will have a common point. There exists a set Z of diameter not exceeding an arbitrarily prescribed positive number ϵ, since we may determine a set X_ϵ in accordance with the assumed condition and may take Z as the set corresponding to the open set complementary to X_ϵ: the relation $|f(x) - f(y)| < \epsilon$ for all x and y outside X_ϵ implies that the diameter of Z does not exceed ϵ. Any such set Z, being closed and bounded, is compact. Hence there exists a unique real number ζ_∞ common to all the sets Z. We now put $f^*(x) = f(x)$ for x in X and $f^*(x_\infty) = \zeta_\infty$. Obviously the function so defined is continuous at every point of X. To show that it is continuous at x_∞, we prescribe $\epsilon > 0$ arbitrarily, determine a corresponding set $X_{\epsilon/2}$ by virtue of the assumed condition, and take Z as the set associated with the open set $U_{\epsilon/2}^*$ complementary to $X_{\epsilon/2}$. Obviously there is a point y in $U_{\epsilon/2}^*$ and in X such that $|f(y) - \zeta_\infty| < \epsilon/2$; and at the same time $|f(x) - f(y)| < \epsilon/2$ for all x and y in $U_{\epsilon/2}^*$. Hence $|f^*(x) - f^*(x_\infty)| = |f(x) - \zeta_\infty| < \epsilon$ for all x in $U_{\epsilon/2}^*$ and in X. Consequently f^* is continuous at x_∞, as we desired to show.

Since the extension of a function f satisfying the given condition is uniquely determined, it is clear that the study of the continuous functions on X which do satisfy the condition is equivalent to the study of the continuous functions on X^*. Hence if we denote the totality of such functions as $\mathfrak{X}$ and the totality of their extensions by $\mathfrak{X}^*$, the approximation theorems in $\mathfrak{X}$ are translatable into approximation theorems in $\mathfrak{X}^*$ and vice versa. In making the indicated translation it is frequently convenient to have a characterization of those functions in $\mathfrak{X}$ which vanish at x_∞ in the sense that $f^*(x_\infty) = 0$. It is easily seen, as a matter of fact, that the property in question is equivalent to the following property: corresponding to $\epsilon > 0$ there is a closed compact subset X_ϵ of X such that $|f(x)| < \epsilon$ for all x outside X_ϵ. The totality of such functions is obviously a closed linear lattice ideal and a closed ring ideal in $\mathfrak{X}$, as we see by directly applying the results of §4 to $\mathfrak{X}^*$. We shall designate this class of functions as $\mathfrak{X}_\infty$. One of the most useful and typical approximation theorems for a locally compact, but not compact, space X is then stated as follows.

Theorem 12: *Let X be a locally compact, but not compact, space; and let $\mathfrak{X}$ and $\mathfrak{X}_\infty$ have the significance indicated above. If $\mathfrak{X}_0$ is any subfamily of $\mathfrak{X}_\infty$ which is a separating family for X and which contains for any x_0 in X a function f_0 such that $f_0(x_0) \neq 0$, then any function in $\mathfrak{X}_\infty$ can be uniformly approximated by linear lattice polynomials or by ordinary polynomials in members of $\mathfrak{X}_0$; and any function in $\mathfrak{X}$ can be similarly approximated by functions which result from the addition of a fixed constant function to such polynomials.*

Proof: If we look at $\mathfrak{X}^*$ and $\mathfrak{X}^*_\infty$, we see that $\mathfrak{X}^*_0$ is a separating family for X^* and that every function f^* in it vanishes at x_∞, $f^*(x_\infty) = 0$. By Theorems 3 and 7 we see that every function in $\mathfrak{X}^*_\infty$ can be uniformly approximated by linear-lattice polynomials in members of $\mathfrak{X}^*_0$. Similarly by Corollary 2 to Theorem 5 we obtain the corresponding statement for ordinary polynomials. If these results are now put in terms of $\mathfrak{X}$, $\mathfrak{X}_\infty$, and $\mathfrak{X}_0$, we obtain the present theorem. If f is in $\mathfrak{X}$, then we see that the function g defined by putting $g(x) = f(x) - f^*(x_\infty)$ for every x is in $\mathfrak{X}_\infty$ since $g^*(x_\infty) = f^*(x_\infty) - f^*(x_\infty) = 0$. Since g can be uniformly approximated by linear-lattice polynomials or ordinary polynomials in members of $\mathfrak{X}_0$, we conclude that f can be approximated by functions obtained by adding a constant function (everywhere equal to $f^*(x_\infty)$) to such a polynomial.

A useful special instance of this theorem may be phrased as follows.

Corollary 1: Let X be a locally compact, but not compact, Hausdorff space; and let $\mathfrak{X}_0$ be a family of real continuous functions on X with the following properties: each function in $\mathfrak{X}_0$ vanishes outside a corresponding compact subset of X; corresponding to any point x_0 in X and any open set U which contains x_0 is a function in $\mathfrak{X}_0$ vanishing outside U and assuming at x_0 a value different from 0. Then $\mathfrak{X}_0$ has all the properties listed in the theorem above.

Proof: Since any compact subset of X is now necessarily closed, it is evident that $\mathfrak{X}_0$ is part of $\mathfrak{X}_\infty$. It is easily verified that $\mathfrak{X}_0$ is a separating family for X, since when x_0 and x_0' are distinct points in X there is an open set U containing x_0 but not x_0' and hence a function f_0 in $\mathfrak{X}_0$ vanishing outside U and not vanishing at x_0, whence $f_0(x_0) \neq f_0(x_0') = 0$. We now see that $\mathfrak{X}_0$ satisfies the requirements laid down in Theorem 12, and the corollary is established.

For the case of complex-valued functions all the preceding remarks can evidently be repeated almost verbatim; the only essential changes which have to be made are to suppress all references to lattice-properties and to require of $\mathfrak{X}_0$ in Theorem 12 and the corollary that it contain $\bar{f}$ along with f. No further comment on this case would seem to be necessary.

7. *The Lebesgue-Urysohn Extension Theorem.* As a first application of the approximation theorems developed in the preceding sections we shall discuss a variant of the celebrated and important Lebesgue-Urysohn extension theorem, which asserts that corresponding to any continuous real function defined on a closed subset X_0 of a normal space X there exists a continuous real function defined on the entire space X and agreeing throughout X_0 with the given function. Since the known proofs of this theorem are quite simple (see, for example, Alexandroff-Hopf [5]), the present discussion is chiefly of interest in the realm of systematics. It would take us too far afield, in any event, to prove the extension theorem in its full generality, since the method to be applied consists in first compactifying the normal space X and then applying the results which we shall establish below. We shall therefore postpone the presentation of the full proof to some other occasion. Here we shall consider the case where X_0 is compact and X arbitrary, obtaining a result which in some respects actually goes beyond that summarized above as the Lebesgue-Urysohn extension theorem. We confine ourselves to the case of real functions, since the complex case is an essentially trivial consequence of it. Accordingly the theorem to be proved here can be stated as follows.

Theorem 13. *Let X be an arbitrary topological space, X_0 a compact subset of X, f_0 a continuous function defined on X_0, and $\mathfrak{X}_0$ a family of continuous real functions defined on X with the following properties: $\mathfrak{X}_0$ is closed under the linear lattice [linear ring] operations; $\mathfrak{X}_0$ is closed under uniform passage to the limit; $\mathfrak{X}_0$ contains all constant functions. In the case of the linear ring operations, let $\mathfrak{X}_0$ have further the property (verified automatically in the linear lattice case) that whenever it contains a function f it also contains a function g which coincides with f on X_0 and has the same bounds on X as f does on X_0. Then for f_0 to be extensible in $\mathfrak{X}_0$, in the sense that there exists a function f in $\mathfrak{X}_0$ agreeing with f_0 throughout X_0, it is necessary and sufficient that f_0 satisfy every linear condition of the form $f(x_0) = f(y_0)$, $x_0 \neq y_0$, $x_0 \in X_0$, $y_0 \in X_0$, which is satisfied by every member of $\mathfrak{X}_0$. The extension f can be so chosen as to have the same bounds on X as f_0 has on X_0.*

Proof: We first remark that the application of the linear lattice [linear ring] operations to extensible functions defined on X_0 produces extensible functions, since $\mathfrak{X}_0$ is assumed to be a linear lattice [linear ring]. We then observe that the uniform limit on X_0 of extensible functions is extensible. Indeed, let a sequence of functions g_n in $\mathfrak{X}_0$ converge uniformly on X_0 to a limit function g_0 (defined on X_0). We may suppose (by thinning out the originally given sequence, should that be necessary) that $|g_{n+1}(x) - g_n(x)| \leq 2^{-n}$ for every x in X_0. We then take h_n to be an extension of $g_{n+1} - g_n$ such that $|h_n| \leq 2^{-n}$. This is always possible, since we can choose h_n as a member of $\mathfrak{X}_0$ which has on X the same bounds as does $g_{n+1} - g_n$ on X_0; or, in the linear lattice case, we can simply take h_n so that

$$h_n(x) = \max(\min(g_{n+1}(x) - g_n(x),\ 2^{-n}),\ -2^{-n}).$$

Now the series $g_1 + \sum_{n=1}^{\infty} h_n$ converges uniformly to a continuous real function g in $\mathfrak{X}_0$ which agrees on X_0 with g_0. The family of all extensible functions on X_0 thus constitutes a closed linear sublattice [linear subring] of the linear lattice [linear ring] of all continuous real functions on X_0, and includes the constant functions on X_0. Theorem 3 and its Corollary in the lattice case, and Theorem 5 and its Corollary 1 in the ring case, show that the family of all extensible functions on X_0 coincides with the family of all those continuous functions on X_0 which satisfy every linear condition of the form $f(x_0) = f(y_0)$, $x_0 \neq y_0$, $x_0 \in X_0$, $y_0 \in Y_0$, which is satisfied by every function in $\mathfrak{X}_0$. The final statement of the theorem is obvious. In the lattice case, if α and β are the greatest lower and least upper bounds of f_0 on X_0 (both finite because X_0 is compact) and if f is any extension of f_0 in $\mathfrak{X}_0$, then f can be replaced by g where $g(x) = \max(\min f(x),\ \beta),\ \alpha)$.

The most interesting case of Theorem 13 is that in which $\mathfrak{X}_0$ is taken to be the family of all continuous functions on X. For this case we may state Theorem 13 in the following form.

Corollary 1. *In order that a continuous real function f_0 defined on a compact subset X_0 of a topological space X have a continuous extension defined over X, it is necessary and sufficient that f_0 satisfy every condition of the form $f(x_0) = f(y_0)$, $x_0 \neq y_0$, $x_0 \in X_0$, $y_0 \in Y_0$ which is satisfied by all continuous functions on X — in other words, that f_0 be constant on every subset of X_0 where all the functions continuous on X are constant by virtue of the topological structure of X. In particular, this condition is superfluous*

when the continuous functions on X constitute a separating family for X_0: every real function continuous on X_0 then has a continuous extension defined on X. If a function f_0 on X_0 has a continuous extension on X then it has such an extension with the same bounds on X as f_0 has on X_0.

8. *The Theorem of Dieudonné.* A second interesting and useful application, still in the field of general topology, can now be made to a situation first adequately discussed by Dieudonné. Here we must presuppose the rudiments of the theory of the cartesian product of topological spaces. It is convenient to think of the product of the spaces X_α (where α runs over a fixed index-set A) as a coördinate space, each point x being specified by its coordinates x_α in the respective factor-spaces X_α. Now if α is a fixed index and f_α is a continuous real function defined on the factor space X_α, we can define a continuous real function f on the product space by putting $f(x) = f_\alpha(x_\alpha)$ where x_α is the coördinate of x corresponding to the index α. Such a function f will be called here a function of one variable — specifically, the function of one variable associated with f_α. These simple preliminaries enable us to state our main result as follows.

Theorem 14: *(Dieudonné, [6]). If X is the cartesian product of compact spaces X_α, $\alpha \in A$, then every continuous real function on X can be uniformly approximated by finite sums of finite products of continuous functions of one variable on X.*

Proof: The cartesian product of compact spaces is known to be compact. Let now $\mathfrak{X}_0$ be the totality of those functions expressible as finite sums of finite products of continuous functions of one variable on X. Since the sums, products, and constant multiples of functions in $\mathfrak{X}_0$ are obviously also in $\mathfrak{X}_0$, we see that $\mathfrak{X}_0$ is a linear subring of the ring $\mathfrak{X}$ of all continuous real functions on X. Obviously $\mathfrak{X}_0$ contains all the constant functions. In order to be able to apply the results of §3 we therefore have to determine what linear relations of the form $f(x) = f(x')$ where $x \neq x'$ are satisfied by every function f in $\mathfrak{X}_0$. Since we may take f here as the function of one variable associated with an arbitrary continuous real function f_α on X_α we must evidently have $f_\alpha(x_\alpha) = f_\alpha(x_\alpha')$ for all f_α. Conversely, if x and x' are points such that $f_\alpha(x_\alpha) = f_\alpha(x_\alpha')$ for every continuous real function f_α on X_α and for all α, then it is evident that $f(x) = f(x')$ for every f in $\mathfrak{X}_0$. Hence we see that any function in $\mathfrak{X}$ which satisfies all the linear relations of the above type can be uniformly approximated by functions in $\mathfrak{X}_0$, by virtue of Theorem 5. We can therefore complete our proof by showing that any continuous real function on X satisfies all these conditions. — First let us consider two points x and x' such that for some fixed index β we have $x_\beta \neq x_\beta'$ while $x_\alpha = x_\alpha'$ for all $\alpha \neq \beta$. From any function f on X we can obtain a function f_β on X_β by putting $f_\beta(y_\beta) = f(y)$ where y_β is arbitrary and $y_\alpha = x_\alpha = x_\alpha'$ for $\alpha \neq \beta$; and f_β is continuous when f is. Hence we see that if x_β and x_β' are points such that $f_\beta(x_\beta) = f_\beta(x_\beta')$ for every continuous real function f_β on X_β, then $f(x) = f_\beta(x_\beta) = f_\beta(x_\beta') = f(x')$. Let us suppose that we have generalized the result just established and have proved that, when x and x' are two points such that for fixed indices $\beta_1, \ldots, \beta_n$ we have $f_{\beta_k}(x_{\beta_k}) = f_{\beta_k}(x'_{\beta_k})$ for every continuous real function f_{β_k} on X_{β_k}, $k = 1, \ldots, n$ while $x_\alpha = x_\alpha'$ for every α other than $\beta_1, \ldots, \beta_n$, then $f(x) = f(x')$ for every f in $\mathfrak{X}$. We can then establish the corresponding result for points differing in at most $n + 1$ coördinates. In

fact let x and x' be given so that $f_{\beta_k}(x_{\beta_k}) = f_{\beta_k}(x'_{\beta_k})$ as above for $k = 1,\ldots,n+1$ while $x_\alpha = x_\alpha'$ for all α other than $\beta_1,\ldots,\beta_{n+1}$. We define a point x'' by putting $x''_{\beta_k} = x_{\beta_k}$ for $k=1,\ldots,n$, $x''_{\beta_{n+1}} = x'_{\beta_{n+1}}$ for $k = n+1$, and $x''_\alpha = x_\alpha = x'_\alpha$ for all α other than $\beta_1,\ldots,\beta_{n+1}$. It is obvious then that $f(x) = f(x'')$ for every f in $\mathfrak{X}$ by the result explicitly proved above. On the other hand the assumption we have made implies that $f(x') = f(x'')$ for every f in $\mathfrak{X}$. Hence we conclude that $f(x) = f(x')$ for every f in $\mathfrak{X}$ as we wished to show. By induction, therefore, we conclude that if two points x and x' differ only in respect to their coördinates for a finite number of indices for each of which $f_\beta(x_\beta) = f(x_\beta')$ for every continuous real function f_β on X_β, then $f(x) = f(x')$ for all f in $\mathfrak{X}$. — Finally let us suppose that f is in $\mathfrak{X}$ and that x and x' are points such that for every α and every continuous real function f_α on X_α the relation $f_\alpha(x_\alpha) = f_\alpha(x_\alpha')$ holds. Then we can determine for any positive ϵ a neighborhood U_ϵ of x such that $|f(x) - f(y)| < \epsilon$ for every y in U_ϵ, by virtue of the continuity of f. By the way in which the topology of X is defined, we can now determine a point x'' which is in U_ϵ and which nevertheless differs from x' only in respect to a finite number of coördinates. This end can indeed be achieved by designating appropriate indices $\beta_1,\ldots,\beta_n$ and putting $x''_{\beta_k} = x_{\beta_k}$ for $k = 1,\ldots,n$ while $x''_\alpha = x'_\alpha$ for all other indices α. We then have $f(x'') = f(x')$ in accordance with our previous results and hence also $|f(x) - f(x')| = |f(x) - f(x'')| < \epsilon$. Since ϵ is arbitrary we conclude that $f(x) = f(x')$, whatever the function f in $\mathfrak{X}$.

Useful variants of this theorem can be obtained by considering the case where some of the factor spaces are locally compact but not compact. As they would involve us in more extensive topological discussions than seem desirable here, we shall leave the matter at this point. — It will be useful, perhaps, to recall a classical application of the theorem just proved: in the theory of integral equations a standard procedure is to replace the kernel $K(x,y)$, assumed to be a continuous function of its arguments on the square $a \le x \le b$, $a \le y \le b$, by a uniformly good approximant of the form $K'(x,y) = F_1(x)G_1(y) + \cdots + F_n(x)G_n(y)$ where $F_1,\cdots,F_n$ and $G_1,\cdots,G_n$ are continuous functions on the interval $[a,b]$. Since the square is the cartesian product of the intervals $a \le x \le b$, $a \le y \le b$, the theorem of Dieudonné gives a direct justification for this device.

9. *The Weierstrass Approximation Theorem.* In the present section we propose to derive from the results of §3 a demonstration of the classical Weierstrass approximation theorem. In spite of the fact that we shall give a comparatively broad version of the theorem, everything we shall have to say is merely a direct specialization of previously established results to the case at hand. The steps of the general development which would have to be retained in a direct independent proof of the Weierstrass theorem will be indicated after the derivation of the theorem from §3 has been presented.

Theorem 15. *(Weierstrass, [7]). Let X be an arbitrary bounded closed subset of n-dimensional cartesian space, the coördinates of a general point being $x_1,\ldots,x_n$. Any continuous real function f defined on X can be uniformly approximated on X by polynomials in the variables $x_1,\ldots,x_n$. In case X contains the origin $x = (0,\ldots,0)$, the function f can be uniformly approximated by polynomials vanishing at the origin if and only if f itself vanishes at the origin. Otherwise f can be uniformly approximated by such polynomials without qualification.*

Proof: The functions $f_1, \ldots, f_n$, where $f_k(x) = x_k$, are continuous real functions of x. They constitute a separating family $\mathfrak{X}_0$ for X since $x = x'$ if and only if $x_k = f_k(x) = f_k(x') = x'_k$ for $k = 1, \ldots, n$. When X does not contain the origin, then we cannot have $f_1(x) = f_n(x) = 0$ for any x in X; but when X contains the origin we obtain $f_1(x) = \ldots = f_n(x) = 0$ by taking x as the origin. Since X is bounded and closed it is compact. Accordingly, the results of §3, particularly those stated in Corollary 2 to Theorem 5, show that any continuous real function f on X can be uniformly approximated by functions of the form (where $a_1, \ldots, a_n$ are positive integers or 0)

$$p(x) = \sum_{1 \leq a_1 + \cdots + a_n \leq N} C_{a_1 \ldots a_n} (f_1(x))^{a_1} \ldots (f_n(x))^{a_n}$$
$$= \sum_{1 \leq a_1 + \cdots + a_n \leq N} C_{a_1 \ldots a_n} x_1^{a_1} \ldots x_n^{a_n},$$

with the proviso that when X contains the origin f must vanish there. If the constant function everywhere equal to 1 is adjoined to the family $\mathfrak{X}_0$, we see that f can be uniformly approximated on X by functions of the form

$$p(x) = \sum_{0 \leq a_1 + \cdots + a_n \leq N} C_{a_1 \ldots a_n} x_1^{a_1} \ldots x_n^{a_n}$$

This completes the proof.

If one wishes to give a direct proof of the Weierstrass approximation theorem by the present methods, the following procedure is available. It is best to consider first the family $\mathfrak{X}_0$ of all homogeneous linear functions, l, where $l(x) = c_1x_1 + \cdots + c_nx_n$, noting that for any given continuous real function f on X (provided f vanishes at the origin when this point is in X) a function l can be found so that $l(x) = f(x)$, $l(y) = f(y)$ at arbitrarily prescribed points x, y in X. Considerations like those used in the proof of Theorem 1 show that such a function f can be uniformly approximated by lattice combinations of functions in $\mathfrak{X}_0$. Theorem 4 has to be established exactly as in §3. It can then be used to convert approximation by lattice combinations into approximation by linear-ring combinations of functions in $\mathfrak{X}_0$, just as was done in the proof of Theorem 5. The main part of the proof is thereby completed. Remarks similar to those above have to be added concerning the adjunction of constant functions to $\mathfrak{X}_0$.

10. *Trigonometric Approximation.* A surprisingly direct and simple application of Theorem 5 yields the fundamental theorem on trigonometric approximation in the real domain, reading as follows.

Theorem 16. *Let f be an arbitrary continuous real function of the real variable θ, $0 \leq \theta \leq 2\pi$, subject to the periodicity condition $f(0) = f(2\pi)$. Then f can be uniformly approximated on its domain of definition by trigonometric polynomials — that is, by functions of the form*

$$p(\theta) = a_0/2 + \sum_{n=1}^{N} (a_n \cos n\theta + b_n \sin n\theta).$$

Proof: It is convenient to make the application of Theorem 5 directly to the case of the unit circle X (given by the equation $x_1^2 + x_2^2 = 1$) in the cartesian plane, thus obtaining a special instance of the Weierstrass approximation theorem as stated in Theorem 15. If we rephrase this special case in terms of the central angle θ corresponding to a general point x on the circle X, we see that the functions continuous on the circle are the functions continuous and periodic in θ

and that the polynomials in x_1 and x_2 (in terms of which such functions can be uniformly approximated) are functions of the form

$p(\theta) = \sum_{1 \leq m+n \leq N} c_{mn} \cos^m \theta \sin^n \theta$ (since $x_1 = \cos\theta$ and $x_2 = \sin\theta$). The addition formulas for the trigonometric functions yield the relations

$$2\cos m\theta \cos n\theta = \cos (m+n)\theta + \cos (m-n)\theta$$

$$2\cos m\theta \sin n\theta = \sin (m+n)\theta + \sin (m-n)\theta$$

$$2\sin m\theta \sin n\theta = -\cos (m+n)\theta + \cos (m-n)\theta$$

which enable us to extablish by a recursive argument that $\cos^m\theta \sin^n\theta$ (and hence also every function of the form $p(\theta)$ described above) is a trigonometric polynomial in the sense required by the statement of the theorem. Indeed, we have only to note that, if $\cos^m \theta \sin^n \theta$ is such a polynomial, then $\cos^{m+1}\theta \sin^n\theta$ and $\cos^m\theta \sin^{n+1}\theta$ are also trigonometric polynomials by virtue of the indicated relations.

The complex form of the trigonometric approximation theorem can be deduced even more readily from Theorem 10, Corollary 2; it can be stated as follows.

Theorem 17. *If f is a continuous complex function of the real variable θ, $0 \leq \theta \leq 2$, subject to the periodicity condition $f(0) = f(2\pi)$ — then f can be uniformly approximated by functions p of the form $p(\theta) = \sum_{n=-N}^{n=+N} C_n e^{ni\theta}$, where the constants C_n are complex numbers.*

Proof: The functions considered will be treated as continuous functions on the unit circle X, θ being the central angle as in the discussion of the preceding theorem. Since $e^{mi\theta} e^{ni\theta} = e^{(m+n)i\theta}$ and $\overline{e^{ni\theta}} = e^{-ni\theta}$, it is clear that the family $\mathfrak{X}_0$ of all functions p of the form $p(\theta) = \sum_{n=-N}^{n=N} C_n e^{ni\theta}$ is a linear subring of the family $\mathfrak{X}$ of all continuous complex functions on X, the function $\overline{p}$ being in $\mathfrak{X}_0$ whenever p is. It is also evident that the functions $e^{ni\theta}$ satisfy no linear relation of the form $e^{ni\theta_0} = 0$ or of the form $e^{ni\theta_1} = e^{ni\theta_2}$, $\theta_1 \neq \theta_2$, where $0 \leq \theta_1 < 2\pi$, $0 \leq \theta_2 < 2\pi$. Consequently, every function in $\mathfrak{X}$ can be uniformly approximated by functions in $\mathfrak{X}_0$, in accordance with Corollary 2 to Theorem 10. This completes the proof.

11. *Approximation by Laguerre Functions.* An important problem of analysis concerns the approximation of real functions continuous on the half-infinite interval $0 \leq x < +\infty$ by linear combinations of the functions $e^{-\alpha x} x^n$, $n = 0,1,2,\ldots$, where α is a fixed positive number. In order to obtain a solution to this problem within the scope of the present discussion we need a lemma concerning the exponential function.

Lemma 1. *On the interval $0 \leq x < \infty$ the function $e^{-n\alpha x}$, where n is a positive integer and $\alpha < 0$, can be uniformly approximated by functions of the form $e^{-\alpha x} p(x)$ where p is a polynomial.*

Proof. We may suppose without loss of generality that $\alpha = 1$. Indeed, if for $\epsilon < 0$ we have found a polynomial $q(x)$ such that $|e^{-nx} - e^{-x} q(x)| < \epsilon$ for $0 \leq x < \infty$, we can replace x by αx and $q(x)$ by the polynomial $p(x) = q(\alpha x)$, obtaining $|e^{-n\alpha x} - e^{-\alpha x} p(x)| < \epsilon$ for $0 \leq x < \infty$. We shall now proceed recursively. When $n = 1$ there is nothing to prove, since e^{-x} is already of the form

specified for its approximants. When $n = 2$, we obtain the desired result by estimating the magnitude of the function f where $f(x) = e^{-2x} - e^{-x} \sum_{k=0}^{N} \frac{(-x)^k}{k!}$. Since f is continuous and has the properties $f(0) = 0$ and $\lim_{x\to\infty} f(x) = 0$, it has an extremum on the interval $0 < x < \infty$. If such an extremum occurs at $x = x_0$, then
$$0 = f'(x_0) = -2e^{-2x_0} + 2e^{-x_0} \sum_{k=0}^{N-1} \frac{(-x_0)^k}{k!} + e^{-x_0} \frac{(-x_0)^N}{N!}$$
so that $f(x_0) = \frac{1}{2} e^{-x_0} \frac{(-x_0)^N}{N!}$. Consequently we have
$\sup_{0\le x<\infty} |f(x)| \le \frac{1}{2N!} \sup_{0\le x<\infty} e^{-x} x^N$. For $N \ge 1$ the function g defined by putting $g(x) = e^{-x} x^N$ is continuous and non-negative on the interval $0 \le x < \infty$ and has the properties $g(0) = 0$, $\lim_{x\to\infty} g(x) = 0$. It therefore has a maximum on the interval $0 < x < \infty$, occurring at the only solution there of the equation $0 = g'(x) = e^{-x} (N - x) x^{N-1}$ — that is, at $x = N$. Accordingly we have $0 \le g(x) \le e^{-N} N^N$. Applying so much of Stirling's formula as is necessary to show that $N! \le e^{-N} N^{N+\frac{1}{2}}/2K$ for a suitable constant $K > 0$ we obtain the inequality $|f(x)| \le KN^{-\frac{1}{2}}$. The case $n = 2$ is thereby settled. If we have proved the lemma for any particular positive integer n, we can discuss the approximation of $e^{-(n+1)x}$ in the following manner. If $\epsilon > 0$ is given, we first use what is known about the approximation of e^{-2ax}, taking $2a = n+1$, so as to obtain a polynomial q such that $|e^{-(n+1)x} - e^{-nx/2 - x/2} q(x)| < \frac{1}{2}\epsilon$. The function $e^{-x/2} |q(x)|$ is bounded on the interval $0 \le x < \infty$. If its least upper bound is A, we use what is known about the approximation of e^{-nax}, taking $a = \frac{1}{2}$, so as to obtain a polynomial r such that $|e^{-nx/2} - e^{-x/2} r(x)| < \epsilon/2A$. It follows that $|e^{-nx/2 - x/2}q(x) - e^{-x}q(x)r(x)| < e^{-x/2} |q(x)| \epsilon/2A \le \frac{1}{2}\epsilon$ and $|e^{-(n+1)x} - e^{-x}p(x)| < \epsilon$, where $p = qr$. Mathematical induction therefore serves to complete the proof of the lemma.

A direct application of Theorem 12 together with the lemma just proved yields the main approximation theorem of this section.

Theorem 18. *Any continuous real function f which is defined on the interval $0 \le x < \infty$ and vanishes at infinity in the sense that $\lim_{x\to\infty} f(x) = 0$ can be uniformly approximated by functions of the form $e^{-ax}p(x)$ where $p(x)$ is a polynomial.*

Proof: We let X be the interval $0 \le x < \infty$. As a topological space, X is locally compact. The function e^{-ax} is in $\mathfrak{X}_\infty$ since $\lim_{x\to\infty} e^{-ax} = 0$. We now let $\mathfrak{X}_0$ consist of this function alone. It is obvious from the monotonicity of the exponential function that $\mathfrak{X}_0$ is a separating family for X. Moreover there is no $x \ge 0$ for which $e^{-ax} = 0$. By hypotheses the function f to be approximated is in $\mathfrak{X}_\infty$. Theorem 12 thus shows that f can be uniformly approximated on X by functions of the form $\sum_{n=1}^{N} C_n e^{-nax}$. Lemma 1 then yields the present theorem.

A variant of the above proof can be based on a direct appeal to the Weierstrass approximation theorem. We introduce a new variable $\xi = e^{-\alpha x}$, $0 < \xi \leq 1$. The function ϕ defined by putting $\phi(0) = 0$, $\phi(\xi) = f(x) = f(\frac{-1}{\alpha} \log \xi)$ is continuous on the interval $0 \leq \xi \leq 1$. Hence ϕ can be uniformly approximated by polynomials $\sum_{n=1}^{N} C_n \xi^n$; and f can be uniformly approximated by functions of the form $\sum_{n=1}^{N} C_n e^{-n\alpha\xi}$. Lemma 1 is then used to complete the proof.

It is of some interest to apply the approximation theorem just proved to derive results concerning approximation in the mean. The classical theorem on this subject reads as follows.

Theorem 19. *The functions of the form $e^{-\alpha x}p(x)$ where $p(x)$ is a polynomial are dense in the function-space $\mathfrak{L}_r(0,\infty)$, $r \geq 1$.*

Proof. Here $\mathfrak{L}_r(0,\infty)$ is the class of all real Lebesgue-measurable functions f on the interval $0 \leq x < \infty$ for which the Lebesgue integral $\int_0^\infty |f(x)|^r dx$ exists. The expression $\left[\int_0^\infty |f(x)|^r dx\right]^{1/r}$ is taken as the norm of f, and $\mathfrak{L}_r(0,\infty)$ then becomes a complete normed linear vector space. It is obvious that every function of the form $e^{-\alpha x}p(x)$, where p is a polynomial, is a member of $\mathfrak{L}_r(0,\infty)$. We wish to prove that if $\epsilon > 0$ and if f is in $\mathfrak{L}_r(0,\infty)$ then there exists a function of this special form for which $\left[\int_0^\infty |f(x) - e^{-\alpha x}p(x)|^r dx\right]^{1/r} < \epsilon$. It is well known that there exists a function g, continuous on the interval $0 \leq x < \infty$ and vanishing outside some bounded interval, for which $\left[\int_0^\infty |f(x) - g(x)|^r dx\right]^{1/r} < \frac{1}{2}\epsilon$. We therefore need only show that it is possible to find a function of the indicated special form for which $\left[\int_0^\infty |g(x) - e^{-\alpha x}p(x)|^r\right]^{1/r} < \frac{1}{2}\epsilon$. The function $e^{\frac{1}{2}\alpha x}g(x)$ is continuous on $0 \leq x < \infty$ and vanishes outside some bounded interval. Hence we can find a polynomial $p(x)$ such that $|e^{\frac{1}{2}\alpha x}g(x) - e^{-\frac{1}{2}\alpha x}p(x)| < \eta$ where $\eta < \frac{1}{2}\left(\frac{r\alpha}{2}\right)^{1/r}\epsilon$. Accordingly we obtain the relations

$$\left[\int_0^\infty |g(x) - e^{-\alpha x}p(x)|^r\right]^{1/r} = \left[\int_0^\infty |e^{\frac{1}{2}\alpha x}g(x) - e^{-\frac{1}{2}\alpha x}p(x)|^r e^{-\frac{r\alpha x}{2}} dx\right]^{1/r} \leq$$

$$\eta\left[\int_0^\infty e^{-\frac{r\alpha x}{2}} dx\right]^{1/r} = \left[\frac{2}{r\alpha}\right]^{1/r} \eta < \tfrac{1}{2}\epsilon.$$

The proof is thereby completed.

A useful related theorem is the following.

Theorem 20. *If f is in $\mathfrak{L}_r(0,\infty)$, $r \geq 1$, or if f is a bounded Lebesgue-measurable function, then the integrals $\int_0^\infty f(x)e^{-\alpha x}x^n dx$ exist for $\alpha > 0$ and $n = 0,1,2,\cdots$. If these integrals vanish for fixed α and $n = 0,1,2,\cdots$, then $f(x)$ vanishes almost everywhere.*

Proof. When f is in $\mathfrak{L}_1(0,\infty)$, the fact that the functions $e^{-\alpha x}x^n$ are bounded yields the existence of the integrals in question. When f is in $\mathfrak{L}_r(0,\infty)$ for $r > 1$, the fact that the functions $e^{-\alpha x}x^n$ are in $\mathfrak{L}_{r'}(0,\infty)$ where $1/r + 1/r' = 1$, yields a like result in the standard way. Finally when f is bounded and Lebesgue-measurable, it is the fact that the functions $e^{-\alpha x}x^n$ are in $\mathfrak{L}_1(0,\infty)$ which yields the desired result. Using this result we see that in every case the function $g(x) = e^{-\frac{1}{4}\alpha x}\int_0^x f(t)e^{-\frac{1}{4}\alpha t}dt$ is a continuous function with the property that $|g(x)| \leq Ke^{-\frac{1}{4}\alpha x}$ for some constant K. Thus g is in $\mathfrak{L}_2(0,\infty)$. Moreover an integration by parts shows that

$$\begin{aligned}\int_0^\infty g(x)e^{-\frac{1}{2}\alpha x}x^n dx &= \int_0^\infty \left(\int_0^x f(t)e^{-\frac{1}{4}\alpha t}dt\right)\left(e^{-\frac{3}{4}\alpha x}x^n\right)dx\\ &= \int_0^\infty \left(f(x)e^{-\frac{1}{4}\alpha x}\right)\left(\int_0^x e^{-\frac{3}{4}\alpha t}t^n dt\right)dx\\ &= \int_0^\infty f(x)e^{-\alpha x}p(x)dx = 0\end{aligned}$$

since $\int_0^x e^{-\frac{3}{4}\alpha t}t^n dt = e^{-\frac{3}{4}\alpha x}p(x)$ where $p(x)$ is a polynomial and since it is assumed that $\int_0^\infty f(x)e^{-\alpha x}x^n dx = 0$ for $n = 0,1,2,\cdots$. We have thus reduced the proof of the theorem to the special case where the given function is in $\mathfrak{L}_2(0,\infty)$. Choosing a polynomial $p(x)$ such that $\left[\int_0^\infty |g(x) - e^{-\frac{1}{2}\alpha x}p(x)|^2 dx\right]^{1/2} < \epsilon$ and noting that $\int_0^\infty g(x)e^{-\frac{1}{2}\alpha x}p(x)dx = 0$, we have $\int_0^\infty |g(x)|^2 dx = \int_0^\infty \left[g(x) - e^{-\frac{1}{2}\alpha x}p(x)\right] g(x)dx \leq \left[\int_0^\infty |g(x) - e^{-\frac{1}{2}\alpha x}p(x)|^2 dx\right]^{1/2} \left[\int_0^\infty |g(x)|^2 dx\right]^{1/2}$ by Schwarz's inequality; and we therefore have $\left[\int_0^\infty |g(x)|^2 dx\right]^{1/2} < \epsilon$. It follows that $\int_0^\infty |g(x)|^2 dx = 0$ and that $g(x)$, being continuous, vanishes identically. The relation $\int_0^x f(t)e^{-\frac{1}{4}\alpha t}dt = 0$ is thus established. From this it follows that $f(x)e^{-\frac{1}{4}\alpha x} = 0$ and $f(x) = 0$, almost everywhere.

In the three theorems proved in this section, it is obvious that the hypotheses concerning the function f can be altered to allow f to be complex, without changing the conclusions.

12. *Approximation by Hermite Functions.* The methods of the preceding section can be applied with little modification to yield comparable results concerning uniform approximation by linear combinations of the functions $e^{-\alpha^2 x^2}x^n$ on the full infinite interval $-\infty < x < +\infty$. Using Theorem 12 and Lemma 1, we immediately obtain the chief result.

Theorem 21. *Any continuous real function f which is defined on the interval $-\infty < x < +\infty$ and which vanishes at infinity in the sense that $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0$ can be uniformly approximated by functions of the form $e^{-\alpha^2 x^2}p(x)$*

where $p(x)$ is a polynomial.

Proof: We let X be the interval $-\infty < x < +\infty$. As a topological space, X is locally compact. The functions f, $e^{-\alpha^2x^2}$, and $e^{-\alpha^2x^2}x$ are in $\mathfrak{X}_\infty$ since they all vanish at infinity in the sense indicated in the statement of the theorem. The family $\mathfrak{X}_0$ consisting of the two functions $e^{-\alpha^2x^2}$, $e^{-\alpha^2x^2}x$ is obviously a separating family for X; and moreover there is no x for which $e^{-\alpha^2x^2} = 0$. Theorem 12 thus shows that f can be uniformly approximated by functions of the form $\sum_{m=1}^{M}\sum_{n=1}^{N} c_{mn}e^{-m\alpha^2x^2}x^n$. To complete the discussion we make use of Lemma 1. Letting A be the maximum of the function $e^{-\frac{1}{2}\alpha^2x^2}|x|^n$, we find a polynomial q such that $|e^{-(2m-1)t}-e^{-t}q(t)| < \epsilon/A$ for $0 \le t < \infty$. We now put $t = \frac{1}{2}\alpha^2x^2$, multiply both sides of the inequality by $e^{-\frac{1}{2}\alpha^2x^2}|x|^n$, and write $p(x)$ for $x^nq(\frac{1}{2}\alpha^2x^2)$, obtaining

$$|e^{-m\alpha^2x^2}x^n - e^{-\alpha^2x^2}p(x)| < e^{-\frac{1}{2}\alpha^2x^2}|x|^n\,\epsilon/A \le \epsilon.$$

The theorem then follows.

A variant of this proof can be given by appropriate use of the Weierstrass approximation theorem, as stated in §9. We introduce a variable point ξ of the cartesian plane with the coordinates $x_1 = e^{-\alpha^2x^2}$, $x_2 = e^{-\alpha^2x^2}x$. The locus of this point ξ, with the origin adjoined, provides a bounded closed set χ in the plane. The function ϕ defined by putting $\phi(\xi) = f(x)$ when $\xi = (x_1, x_2) = (e^{-\alpha^2x^2}, e^{-\alpha^2x^2}x)$ and $\phi(\xi) = 0$ when ξ is the origin is a continuous function of ξ on X. The Weierstrass approximation theorem shows that ϕ can be uniformly approximated on X by a polynomial $\sum_{l=1}^{L}\sum_{n=1}^{N} c'_{ln}x_1^l x_2^n$. Hence f can be uniformly approximated on the interval $-\infty < x < +\infty$ by a function of the form $\sum_{l=1}^{L}\sum_{n=1}^{N} c'_{ln}e^{-(l+n)\alpha^2x^2}x^n = \sum_{m=1}^{M}\sum_{n=1}^{N} c_{mn}e^{-m\alpha^2x^2}x^n$, where $m = l+n$, $M = L+N$, and $c_{mn} = c'_{m-n,n}$. The remainder of the proof is identical with that given above.

By a method almost the same as that used in proving Theorem 19, we obtain the corresponding result for approximation in the mean by functions of the form $e^{-\alpha^2x^2}p(x)$ where p is a polynomial. The theorem is therefore stated without detailed proof.

Theorem 22. *The functions of the form $e^{-\alpha^2x^2}p(x)$, where $p(x)$ is a polynomial, are dense in the function-space $\mathfrak{L}_r(-\infty,+\infty)$, $r \ge 1$.*

By using the results of Theorem 20 we can give a simple proof of its analogue for the functions $e^{-\alpha^2x^2}x^n$.

Theorem 23. *If f is in $\mathfrak{L}_r(-\infty,+\infty)$, $r \ge 1$, or if f is a bounded Lebesgue-measurable function, then the integrals $\int_{-\infty}^{+\infty} f(x)e^{-\alpha^2x^2}x^n dx$ exist for $\alpha > 0$ and $n = 0,1,2,\cdots$. If these integrals vanish for fixed α and $n = 0,1,2,\cdots$, then $f(x)$ vanishes almost everywhere.*

Proof: Let f_1 and f_2 be the functions defined by putting $2f_1(x) = f(x) + f(-x)$, $2f_2(x) = f(x) - f(-x)$, so that f_1 is even, f_2 is odd, and $f = f_1 + f_2$. It is evident that f_1 and f_2 are in $\mathcal{L}_r(-\infty,+\infty)$ when f is, and are bounded when f is. We easily see that

$$2\int_0^\infty f_2(x)e^{-a^2x^2}x^{2m+1}dx = \int_{-\infty}^{+\infty} f_2(x)e^{-a^2x^2}x^{2m+1}dx = \int_{-\infty}^{+\infty} f(x)e^{-a^2x^2}x^{2m+1}dx = 0.$$

The function f_2, considered on the half-infinite interval $0 \leq x < \infty$, is in $\mathcal{L}_r(0,\infty)$ or is bounded according as f_2 is in $\mathcal{L}_r(-\infty,+\infty)$ or is bounded. We now make the change of variable $t = x^2$ and write $g(t) = f_2(x)e^{-\frac{1}{2}a^2x^2}$, obtaining

$$\int_0^\infty g(t)e^{-\frac{1}{2}a^2t}t^m dt = 2\int_0^\infty f_2(x)e^{-a^2x^2}x^{2m+1}dx = 0,$$

$$\int_0^\infty |g(t)|^r dt = 2\int_0^\infty |f_2(x)|^r e^{-\frac{r}{2}a^2x^2}x dx < +\infty$$

so that g is in $\mathcal{L}_r(0,\infty)$ if f is and is bounded if f is. Thus Theorem 20 is applicable and yields the result that $g(x)$ vanishes almost everywhere. It follows that f is essentially an even function of x, the equation $f(-x) = f(x)$ being satisfied almost everywhere. It is easy to see that the function h defined by putting $h(x) = f(x)e^{-\frac{1}{2}a^2x^2}x$ is essentially odd; that h is in $\mathcal{L}_r(-\infty,+\infty)$ when f is and is bounded when f is; and that $\int_{-\infty}^{+\infty} h(x)e^{-\frac{1}{2}a^2x^2}x^n dx = 0$ for $n = 0,1,2,\cdots$. By what has already been proved h must be essentially even. However, since it was given as an essentially odd function, it must vanish almost everywhere. Thus $f(x) = 0$ almost everywhere, as we wished to prove.

It is obvious that in the three theorems proved in this section the hypotheses concerning the function f can be altered so as to allow f to be complex without changing the conclusions.

13. *The Peter-Weyl Approximation Theorem.* As a final application of our results concerning approximation, we shall sketch briefly a proof of the theorem of Peter and Weyl concerning the approximation of functions on a compact topological group*. This theorem includes as a special case the classical theorems on trigonometric approximation given in §10, as is well-known.

Let X be a compact topological group. It is known that this group has a complete system of mutually inequivalent, irreducible continuous real (respectively, complex) matrix representations by finite orthogonal (respectively, unitary) matrices. Specifically a system of finite orthogonal (respectively, unitary) matrices $\Lambda^{(k)}(x)$, where x is in X and $k = 1,2,3,\cdots$, can be found with the following properties:

(1) the elements of $\Lambda^{(k)}(x)$ depend continuously on x; the relation $\Lambda^{(k)}(x)\Lambda^{(k)}(y) = \Lambda^{(k)}(xy)$ is satisfied; and $\Lambda^{(k)}(e)$, e being the identy element of X, is a unit matrix;

(2) the continuous representation of X given by $\Lambda^{(k)}(x)$ is irreducible;

(3) the continuous representations of X given by $\Lambda^{(k)}(x)$ and $\Lambda^{(k')}(x)$ are inequivalent when $k \neq k'$;

*The proof offered here was presented by the author to a seminar held at the University of Buenos Aires in 1943.

(4) any irreducible continuous real (respectively, complex) matrix representation of X is equivalent to the representation given by $\Lambda^{(k)}(x)$ for some k (necessarily unique);

(5) any two distinct elements x and y of the group X determine at least one k such that $\Lambda^{(k)}(x) \neq \Lambda^{(k)}(y)$.

The problem to be discussed is that of approximating a general continuous function on X in terms of the functions $\lambda_{ij}^{(k)}$ defined by taking $\lambda_{ij}^{(k)}(x)$ as the element standing in the i^{th} row and j^{th} column of the matrix $\Lambda^{(k)}(x)$. We shall therefore take $\mathfrak{X}$ to be the family of all continuous real (respectively, complex) functions on X, and $\mathfrak{X}_0$ as the family comprising all the functions $\lambda_{ij}^{(k)}$, $i = 1,\cdots,n_k$, $j = 1,\cdots,n_k$, $k = 1,2,3,\cdots$. By virtue of (1) we see that $\mathfrak{X}_0$ is part of $\mathfrak{X}$. By virtue of (5) we see that $\mathfrak{X}_0$ is a separating family for X. A trivial irreducible representation of X can be obtained by letting $\Lambda(x)$ be the matrix of one row and one column whose single element has the value 1. From (1) and (4) we must have $\Lambda(x) = \Lambda^{(k)}(x)$ for some k, necessarily unique. Hence $\mathfrak{X}_0$ contains the constant function which assumes the value 1 everywhere on X. In the complex case, it is evident that the matrix $\overline{\Lambda^{(k)}(x)}$ whose elements are the conjugates of the elements of $\Lambda^{(k)}(x)$ gives an irreducible unitary representation of X, by virtue of the relation $\overline{\Lambda^{(k)}(x)}\,\overline{\Lambda^{(k)}(y)} = \overline{\Lambda^{(k)}(xy)}$. Because of (4) this representation is equivalent to that given by the matrices $\Lambda^{(k')}(x)$ for some k, necessarily unique. Thus there is a non-singular constant matrix A such that $\overline{\Lambda^{(k)}(x)} = A^{-1}\Lambda^{(k')}(x)A$. Accordingly each of the functions $\overline{\lambda_{ij}^{(k)}}$ is a linear combination (with constant coefficients) of the functions $\lambda_{i'j'}^{(k')}$, $i' = 1,\cdots,n_k{}'$, $j' = 1,\cdots,n_k{}'$. The most important property of all is expressed by a similar statement — namely, that the product $\lambda_{ij}^{(k)}\lambda_{i'j'}^{(k')}$ is a finite linear combination with real (respectively, complex) constant coefficients of the functions $\lambda_{i''j''}^{(k'')}$. The proof of this assertion is obtained by considering the representation of X given by the Kronecker product $\Lambda^{(k)}(x) \times \Lambda^{(k')}(x)$ of the matrices $\Lambda^{(k)}(x)$ and $\Lambda^{(k')}(x)$. The Kronecker product in question is a matrix of $n_k^2 n_{k'}^2$ elements whose rows and columns are labeled by the pairs (i,i'), (j,j') respectively, the element standing in the row labeled (i,i') and column labeled (j,j') being $\lambda_{ij}^{(k)}(x)\lambda_{i'j'}^{(k')}(x)$. By direct computation it is easy to verify that

$$\left[\Lambda^{(k)}(x) \times \Lambda^{(k')}(x)\right]\left[\Lambda^{(k)}(y) \times \Lambda^{(k')}(y)\right] =$$

$$\left[\Lambda^{(k)}(x)\Lambda^{(k)}(y)\right]\times\left[\Lambda^{(k')}(x)\Lambda^{(k')}(y)\right] = \Lambda^{(k)}(xy) \times \Lambda^{(k')}(xy).$$

Hence the Kronecker product provides a continuous representation of X. When this representation is resolved into its irreducible constituents, finite in

number, each of the latter is equivalent in accordance with (4) to a representation $\Lambda^{(k'')}(x)$ for some k'', necessarily unique. This resolution corresponds to the determination of a matrix Λ of real (respectively, complex) constants such that the matrices $\Lambda\left[\Lambda^{(k)}(x) \times \Lambda^{(k')}(x)\right]\Lambda^{-1} = \Lambda(x)$ have the form indicated schematically as follows:

$$\begin{pmatrix} \Lambda^{(k_1'')}(x) & 0 & 0 & 0 \\ 0 & \Lambda^{(k_2'')}(x) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Lambda^{(k_n'')}(x) \end{pmatrix}$$

Here the blocks along the principal diagonal are occupied by various ones of the matrices $\Lambda^{(k'')}(x)$ (not necessarily distinct!) and all other blocks are filled with zeros. The fact that $\Lambda^{(k)}(x)\Lambda^{(k')}(x) = \Lambda^{-1}\Lambda(x)\Lambda$ leads at once to the conclusion that $\lambda^{(k)}_{ij}\lambda^{(k')}_{i'j'}$ is a real (respectively, complex) linear combination (with constant coefficients) of functions $\lambda^{(k'')}_{i''j''}$ corresponding to elements of the diagonal blocks in the matrix $\Lambda(x)$.

From the properties enumerated above we can now obtain the approximation theorem by direct application of Theorems 5 and 10.

Theorem 24 *(Peter-Weyl, [8]). Any continuous real (respectively, complex) function on the compact topological group X can be uniformly approximated on X by real (respectively, complex) linear combinations of the functions $\lambda^{(k)}_{ij}$ described above in terms of the irreducible representations of X.*

Proof. Let $\mathfrak{U}(\mathfrak{X}_0)$ be the family of all the real (respectively, complex) continuous functions on X which can be uniformly approximated by polynomials in members of $\mathfrak{X}_0$, the family of all the functions $\lambda^{(k)}_{ij}$. In view of the facts about the products $\lambda^{(k)}_{ij}\lambda^{(k')}_{i'j'}$, it is clear that $\mathfrak{U}(\mathfrak{X}_0)$ is also the family of all the real (respectively, complex) continuous functions on X which can be uniformly approximated by real (respectively, complex) linear combinations of the functions $\lambda^{(k)}_{ij}$. In the complex case, we know further that $\mathfrak{U}(\mathfrak{X}_0)$ contains $\bar{f}$ along with f. Since $\mathfrak{X}_0$ is a separating family for X and contains non-vanishing

constant functions, Theorem 5 shows that $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$ in the real case and Theorem 10 leads to the same result in the complex case.

14. *Linear Combinations of Prescribed Functions.* It would be natural to study, by way of further generalizing the results obtained here, the problem of approximation in terms of linear combinations of prescribed functions. In this domain, however, there are encountered some of the most difficult problems of analysis. For example, Wiener [9] has shown that general Tauberian theorems are intimately related to the problem of approximation in the mean by linear combinations of functions f_a obtained from a single function f in $\mathfrak{L}_1(-\infty, +\infty)$ by putting $f_a(x) = f(x - a)$. The conditions under which every function in $\mathfrak{L}_1(-\infty, +\infty)$ can be approximated in the mean (of order one) by such linear combinations were obtained by Wiener with the use of ingenious and powerful methods. Modern versions of his treatment have brought many simplifications, but still leave the impression that the results are among the deeper achievements of analysis.

In general, therefore, one cannot expect that the theory of this broader problem will assume so satisfactory a form as that which has been worked out when the lattice or the ring operations can be used to build approximants. The fact that we have been able in §§10-13 to apply our theory to obtain results concerning particular cases of the broader problem is simply due to the observation that under certain circumstances it is possible to approximate products of the prescribed functions by linear combinations of them. This observation leads to an application of the ring theorems given in §3 and §5, in the manner exemplified in §§10-13. Whenever a special theorem concerning the approximation of products by linear combinations can be established, the way is open for the employment of the same device.

REFERENCES

[1] Stone, M. H., Transactions of the American Mathematical Society, 41 (1937), pp. 375-481, especially pp. 453-481.

[2] Kakutani, Shizuo, Annals of Mathematics, (2) 42 (1941), pp. 994-1024, especially pp. 1004-1005.

[3] Šilov, G., Comptes Rendus (Doklady), Akademia Nauk, U.S.S.R., 22 (1939), pp. 7-10.

[4] Čech, E., Annals of Mathematics, (2) 38 (1937), pp. 823-844.

[5] Alexandroff, P., and Hopf, H., "Topologie I," (Berlin, 1935), pp. 73-78.

[6] Dieudonné, J., Comptes Rendus (Paris), 205 (1937), p. 593.

[7] Weierstrass, Karl, Mathematische Werke, Band 3, Abhandlungen III, pp. 1-37, especially p. 5 (= Sitzungsberichte, Kön. Preussischen Akademie der Wissenschaften, July 9 and July 30, 1885).

[8] Peter, F., and Weyl, H., Mathematische Annalen, 97 (1927), pp. 737-755, especially p. 753.

[9] Wiener, Norbert, "The Fourier Integral," (Cambridge, England, 1933) especially Chapter II, §14.

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THE MINIMUM OF A REAL, INDEFINITE, BINARY QUADRATIC FORM

by Gordon Pall

A simple proof will be given that the minimum m (for integral values not all zero of its variables) of a real, indefinite, binary quadratic form f of discriminant d satisfies $m^2 \leq d/5$, and indeed satisfies $m^2 \leq d/8$ unless f is equivalent to $m(x^2 + xy - y^2)$. This proof is of interest in that it yields simultaneously the first two Markoff minima, and makes no use either of continued fractions, the roots, or the theory of reduction of binary quadratic forms.

Changing f to its negative, if necessary, we can suppose $m > 0$; and, not assuming that f actually attains its minimum, we can write $f = ax^2 + bxy + cy^2$, with $m \leq a \leq m + \epsilon$, for any preassigned ϵ, $0 \leq \epsilon < \frac{1}{2}a$. If x is replaced by $x + hy$, b becomes $b + 2ha$. An integer h can be chosen and the sign of y adjusted so that $0 \leq b \leq a$. Then $c < 0$, since otherwise we would have $c \geq m \geq a - \epsilon$, and $d = b^2 - 4ac \leq a^2 - 4a(a - \epsilon) < 0$. Accordingly, f can be given the notation

(1) $\quad f = ax^2 + bxy - a_1y^2,\ 0 \leq b \leq a \leq m + \epsilon,\ m \leq a,\ m \leq a_1.$

If now $b \geq a_1 - \epsilon$, then

(2) $\quad d = b^2 + 4aa_1 \geq (a_1 - \epsilon)^2 + 4am.$

If $b < a_1 - \epsilon$, then the number $a + b - a_1$ represented by f cannot be positive, since then $a + b - a_1 \geq m \geq a - \epsilon$; hence $a_1 - a - b \geq m$, and

(3) $\quad d = (b + 2a)^2 + 4a(a_1 - a - b) \geq 4a^2 + 4am.$

Letting $\epsilon \to 0$, (2) and (3) together imply that $d \geq 5m$.

Indeed, by (3), $d \geq 8m^2$, unless (1) holds with $b \geq a_1 - \epsilon$ for an infinity of ϵ's near zero. In this case, the fixed number d must have the limiting value $5m^2$ obtained by letting $\epsilon \to 0$ in (1). Also, f is equivalent to a sequence of forms which approach $m(x^2 + xy - y^2)$ as a limit. Thus we can take $a = m + s_1$, $a_1 = m + s_2$, $b = m - s_0$, where s_1 and s_2 are nonnegative numbers which are small with ϵ, and since

(4) $\quad (m - s_0)^2 + 4(m + s_1)(m + s_2) = 5m$,

s_0 is also nonnegative. By (4), $2m(2s_1 + 2s_2 - s_0) = -(s_0^2 + 4s_1s_2)$. Hence, either $s_0 = s_1 = s_2 = 0$, or $s_0 > 2s_1 + 2s_2$. 8n the latter case, f represents $a + b - a_1 = m + s_1 - s_2 - s_0 < m$, a contradiction.

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KUMMER NUMBERS

by P. A. Piza

Let ${}_nD_c$ be integers defined by the following relations:

$$ {}_nD_1 = 1. \qquad {}_nD_{c>n} = 0. \qquad {}_nD_{c<1} = 0. \qquad {}_nD_c = c({}_{n-1}D_c + {}_{n-1}D_{c-1}). $$

A short table of D-numbers is as follows:

n \ c	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(1)	1									
(2)	1	2								
(3)	1	6	6							
(4)	1	14	36	24						
(5)	1	30	150	240	120					
(6)	1	62	540	1560	1800	720				
(7)	1	126	1806	8400	16800	15120	5040			
(8)	1	254	5796	40824	126000	191520	141120	40320		
(9)	1	510	18150	186480	834120	1905120	2328480	1451520	362880	
(10)	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	3628800

D-numbers have been found by the author to possess remarkable properties, among them the following:

$$ x^t - (x-1)^t + \sum_{\alpha=1}^{t} {}_tD_\alpha \binom{x-1}{\alpha-1}. $$

(1)..
$$ x^t = \sum_{\alpha=1}^{t} {}_tD_\alpha \binom{x}{\alpha}. $$

$$ \sum_{\beta=1}^{x} \beta^t = \sum_{\alpha=1}^{t} {}_tD_\alpha \binom{x+1}{\alpha+1}. $$

With this last formula it is possible to compute the sum of the tenth powers of the first one thousand integers, it being the 32-digit number

91,409,924,241,424,243,424,241,924,242,500

in a simpler manner than was done by Jacob Bernoulli in ARS CONJECTAND I with the first use that he made of his famous fractionary Bernoullian Numbers.*

The development of

$$ \sum_{\beta=1}^{1000} \beta^{10} = \sum_{\alpha=1}^{10} \binom{1001}{\alpha+1} {}_{10}D_\alpha $$

is as follows:

*See SOURCE BOOK OF MATHEMATICS page 90.

1	$\binom{1001}{2}$ =	500,500
$+1022$	$\binom{1001}{3}$	170,333,163,000
$+55980$	$\binom{1001}{4}$	2,327,832,672,165,000
$+818520$	$\binom{1001}{5}$	6,786,929,139,064,074,000
$+5103000$	$\binom{1001}{6}$	7,023,889,581,003,395,100,000
$+16435440$	$\binom{1001}{7}$	3,215,573,813,620,290,802,680,000
$+29635200$	$\binom{1001}{8}$	720,412,779,053,311,333,189,200,000
$+30240000$	$\binom{1001}{9}$	81,107,697,233,553,078,668,580,000,000
$+16329600$	$\binom{1001}{10}$	4,344,777,125,406,971,318,118,493,440,000
$+3628800$	$\binom{1001}{11}$	86,983,315,783,400,173,257,685,393,920,000
		91,409,924,241,424,243,424,241,924,242,500

Professor Jekuthiel Ginsburg detected in these D-coefficients the following relation:

(2).. $\quad {}_nD_c = c!\,{}_{n-1}T_{n-c}$,

where the ${}_{n-1}T_{n-c}$ are the Stirling Numbers of the second kind.

The object of this paper is to present another family of very interesting numbers derivable from D-coefficients, which family of numbers Prof. Oystein Ore has called Kummer Numbers in a recent letter to the writer, because he has observed that they happen to be the coefficients of the Kummer polynomials $P_i(x,y)$.*

Let us consider the following numerical instance of (1):

$5^5 = 1\binom{5}{4} + 30\binom{5}{3} + 150\binom{5}{2} + 240\binom{5}{1} + 120\binom{5}{0}$.

By repeated application of the rule of formation of binomial coefficients $\binom{n}{c}=\binom{n-1}{c}+\binom{n-1}{c-1}$, the above numerical function can be successively expressed as follows:

$$
\begin{aligned}
5^5 &= 1\binom{5}{4} + 1\binom{5}{3} + 29\binom{5}{2} + 121\binom{5}{1} + 119\binom{5}{0}\\
&\qquad\quad + 29\binom{5}{3} + 121\binom{5}{2} + 119\binom{5}{1} + 1\binom{5}{0}\\
&= 1\binom{6}{4} + 29\binom{6}{3} + 121\binom{6}{2} + 119\binom{6}{1} + 1\binom{5}{0}\\
&= 1\binom{6}{4} + 1\binom{6}{3} + 28\binom{6}{2} + 93\binom{6}{1}\\
&\qquad\quad + 28\binom{6}{3} + 93\binom{6}{2} + 26\binom{6}{1} + 1\binom{5}{0}\\
&= 1\binom{7}{4} + 28\binom{7}{3} + 93\binom{7}{2} + 26\binom{6}{1} + 1\binom{5}{0}\\
&= 1\binom{7}{4} + 1\binom{7}{3} + 27\binom{7}{2}\\
&\qquad\quad + 27\binom{7}{3} + 66\binom{7}{2} + 26\binom{6}{1} + 1\binom{5}{0}\\
&= 1\binom{8}{4} + 27\binom{8}{3} + 66\binom{7}{2} + 26\binom{6}{1} + 1\binom{5}{0}\\
&= 1\binom{8}{4} + 1\binom{8}{3}\\
&\qquad\quad + 26\binom{8}{3} + 66\binom{7}{2} + 26\binom{6}{1} + 1\binom{5}{0}\\
&= 1\binom{9}{4} + 26\binom{8}{3} + 66\binom{7}{2} + 26\binom{6}{1} + 1\binom{5}{0}\,.\\
5^5 &= 1\binom{9}{5} + 26\binom{8}{5} + 66\binom{7}{5} + 26\binom{6}{5} + 1\binom{5}{5}\,.
\end{aligned}
$$

*See Dickson's History of the Theory of Numbers', volume II, pages 741 and 761, citations 76 and 180. This reference is due to Prof. Ore.

We have thus met the symmetrical set of Kummer Numbers 1, 26, 66, 26, 1, which is a partition of 120 = 5!, in combined products with a *vertical* set of adjacent binomial coefficients, to obtain a partition of the fifth power of an integer.

By similar treatment of the consecutive rows of D-numbers, we get other symmetrical sets of these Kummer Numbers, one such set of t numbers for each exponent t, as follows:

For $t=1$: 1
For $t=2$: 1, 1
For $t=3$: 1, 4, 1
For $t=4$: 1, 11, 11, 1
For $t=5$: 1, 26, 66, 26, 1
For $t=6$: 1, 57, 302, 302, 57, 1 .

Note that the sum of each set is $t!$

Kummer Numbers are defined by the following relations:

$$ {}_tK_1 = 1 = {}_tK_t \ . \qquad {}_tK_{c>t} = 0 = {}_tK_{c<1} \ . $$
$$ {}_tK_c = c\cdot {}_{t-1}K_c + (t+1-c)\cdot {}_{t-1}K_{c-1} \ . $$

A short table of Kummer Numbers is as follows:

t \ c	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(1)	1									
(2)	1	1								
(3)	1	4	1							
(4)	1	11	11	1						
(5)	1	26	66	26	1					
(6)	1	57	302	302	57	1				
(7)	1	120	1191	2416	1191	120	1			
(8)	1	247	4293	15619	15619	4293	247	1		
(9)	1	502	14608	88234	156190	88234	14608	502	1	
(10)	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

As a consequence of the summation relations of D-coefficients, these Kummer Numbers have the following properties:

$$ x^t - (x-1)^t = \sum_{a=1}^{t} {}_tK_a \binom{x\ t-1-a}{t-1} \ . $$
$$ x^t = \sum_{a=1}^{t} {}_tK_a \binom{x+t-a}{t} . $$
$$ \sum_{\beta=1}^{x} \beta^t = \sum_{a=1}^{t} {}_tK_a \binom{x+t+1-a}{t+1} \ . $$

$$\sum_{\beta=1}^{x}\ \sum_{\gamma=1}^{\beta} \gamma^{t} = \Sigma^{2}\ x^{t} = \sum_{\alpha=1}^{t} {}_{t}K_{\alpha} \binom{x+t+2-\alpha}{t+2} ,$$

where the superscript 2 in Σ^2 means that there are to be two successive summations of the t-th powers of the first x integers.

In general, for m successive summations of the powers, we will have

$$(3)\ldots \qquad \Sigma^{m}\ x^{t} = \sum_{\alpha=1}^{t} {}_{t}K_{\alpha} \binom{x+t+m-\alpha}{t+m} ,$$

where m may be any positive or negative integer or zero.

When $m = 0$:

$$\Sigma^{0}\ x^{t} = x^{t} ,$$

and the superscript zero means that no summation nor subtraction is to be performed on x^t.

When $m = -1$:

$$\Sigma^{-1}\ x^{t} = x^{t} - (x-1)^{t},$$

where the superscript $m = -1$ means a negative summation or the difference between x^t and $(x-1)^t$.

Also

$$\Sigma^{-2}\ x^{t} = x^{t} - 2(x-1)^{t} + (x-2)^{t},$$

where the superscript -2 means the difference of the differences between x^t, $(x-1)^t$ and $(x-2)^t$.

$$\Sigma^{-3}\ x^{t} = x^{t} - 3(x-1)^{t} + 3(x-2)^{t} - (x-3)^{t}.$$

$$\Sigma^{-t}\ x^{t} = t!$$

Thus the successive sums and differences of the t-th powers of the integers up to x can be formulated in one general and concise notation and summation (3).

If the Fermat equation $x^t + y^t = z^t$ is ever possible in integers, it can be expressed in terms of our summations, as follows:

$$\sum_{\alpha=1}^{t} {}_{t}D_{\alpha} \left[\binom{x}{\alpha} + \binom{y}{\alpha} \right] = \sum_{\alpha=1}^{t} {}_{t}D_{\alpha} \binom{z}{\alpha} .$$

$$\sum_{\alpha=1}^{t} {}_{t}K_{\alpha} \left[\binom{x+t-\alpha}{t} + \binom{y+t-\alpha}{t} \right] = \sum_{\alpha=1}^{t} {}_{t}K_{\alpha} \binom{z+t-\alpha}{t} .$$

San Juan, Puerto Rico

SEGMENT - FUNCTIONS

by John E. Freund

The purpose of this paper is to construct a function consisting of a finite set of straight line segments with the condition that the function be single valued. At points of discontinuity the function is made single valued by defining: $f(c) = \lim_{x \to c^+} f(x)$

In the construction of this function we shall use two basic functions which

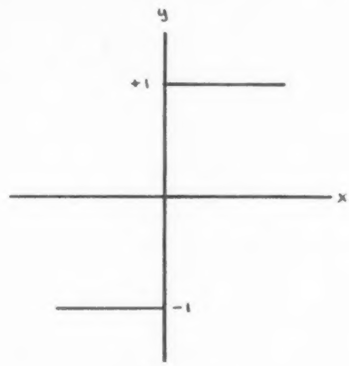


fig. a

are defined as follows. 1) signum of x, written as sgn (x) where:

$$\text{sgn}(x) = \begin{matrix} +1 \text{ when } x > \text{ or } = \text{ to } 0 \\ -1 \text{ when } x < 0 \end{matrix}$$

This function can easily be moved in the plane by suitable linear transformations and the distance between its two parts widened through multiplication by a constant. 2) The other function is defined as: $y = \dfrac{|x| + x}{2}$ for all values of x. The slope of the

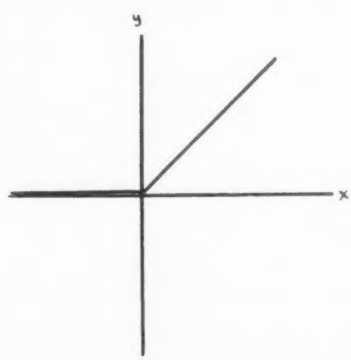


fig. b

slanted part of the curve can be varied, by multiplying by a constant, and the curve can be moved by suitable linear transformations. (fig. b)

Let us first construct a function which consists of one line segment somewhere in the plane and which is zdro elsewhere. (fig. c) It should be noted that at the two points of discontinuity the value of the function is

taken as the value approached from the right according to definition.

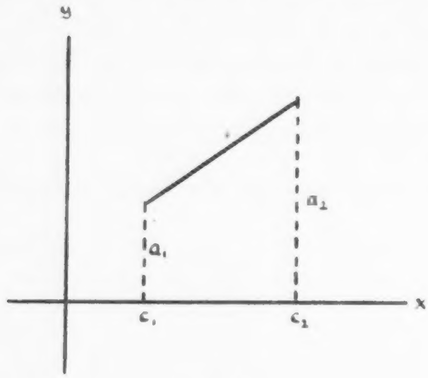


fig. c

This function is constructed by the process of adding ordinates of the four functions given below and shown in fig. d. The equations of these four functions are:

$$y_1 = a_1 + \frac{a_2 - a_1}{c_2 - c_1}\left\{\frac{(x-c_1) + |x-c_1|}{2}\right\}$$

$$y_2 = -a_2 - \frac{a_2 - a_1}{c_2 - c_1}\left\{\frac{(x-c_2) + |x-c_2|}{2}\right\}$$

$$y_3 = -\frac{a_1}{2}[1-\operatorname{sgn}(x-c_1)]$$

$$y_4 = \frac{a_2}{2}[1-\operatorname{sgn}(x-c_2)]$$

These four functions are special cases of the functions 1) and 2) and their graphs are shown in the following diagram.

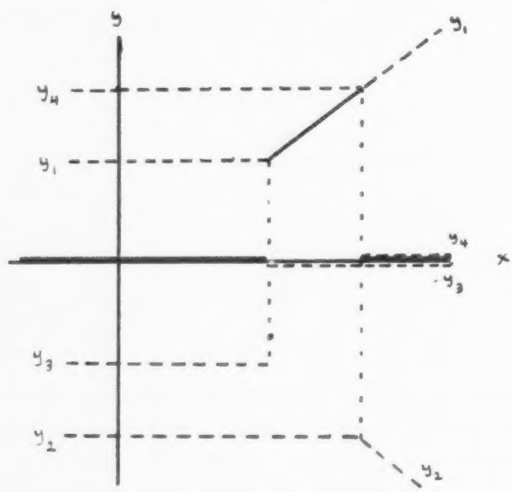


fig. d

The sum of these four functions is given by:

$$(3)\qquad y = \tfrac{1}{2}\left[\frac{a_2 - a_1}{c_2 - c_1}\right]\left\{|x-c_1| - |x-c_2|\right\} + \tfrac{1}{2}\left\{a_1 \operatorname{sgn}(x-c_1) - a_2 \operatorname{sgn}(x-c_2)\right\}$$

It can easily be checked that the sum of these four functions gives us the desired function of fig. c. It contains the line segment and is zero elsewhere. The left end point of the segment is contained in the graph whereas the right end point is taken as zero. This choice is desirable because it avoids difficulties when we add several of these line segments together. To avoid confusion let us define a^+ as the value of the function when approached from the right and a^- when it is approached from the left at possible points of discontinuity.

II.

We can now construct functions containing any finite number of line segments by taking the above function (3) for each segment separately and adding these functions together. There is no interference between these functions because they are a) single-valued, b) do not overlap as each function equals zero except for the specific line segment and c) at points where the segments come together, one of them is defined as zero. If we take any set of points (c_i, a_i), where a_i may take on two values, a_i^+ and a_i^-, the function containing all the line segments connecting successive points, and which is zero for $x < c_0$ and $x > c_n$ (the first and last points), is given by the summation:

$$(4)\qquad y = \tfrac{1}{2}\sum_{i=0}^{n-1}\left\{\frac{a_{i+1}^- - a_i^+}{c_{i+1} - c_i}\right\}\left[|x-c_i| - |x-c_{i+1}|\right] + a_i^+ \operatorname{sgn}(x-c_i) - a_{i+1}^- \operatorname{sgn}(x-c_{i+1})$$

If there are no points of discontinuity except at the ends, the function can be simplified to:

$$(5)\qquad y = \tfrac{1}{2}\left\{a_0 \operatorname{sgn}(x-c_0) - a_n \operatorname{sgn}(x-c_n) + \sum_{i=0}^{n-1}\left\{\frac{a_{i+1} - a_i}{c_{i+1} - c_i}\right\}\left[|x-c_i| - |x-c_{i+1}|\right]\right\}$$

If furthermore the ordinates of the first and last points are zero, the function is given by:

$$(6)\qquad y = \tfrac{1}{2}\sum_{i=0}^{n-1}\left\{\frac{a_{i+1} - a_i}{c_{i+1} - c_i}\right\}\left\{|x-c_i| - |x-c_{i+1}|\right\}$$

III.

As illustrations let us consider the following three simple examples. First the function given in fig. e consisting of a finite number of line segments each of slope 1, and the x coordinates 1 unit apart.

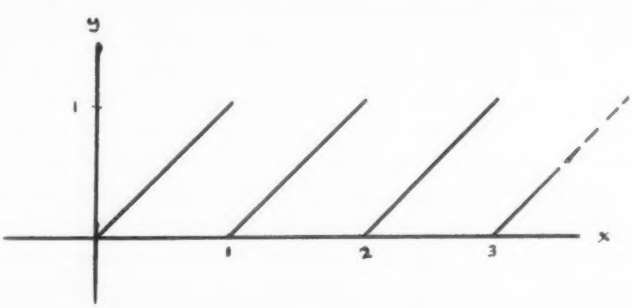


fig. e

From the general formula (3) it can easily be shown that its equation is

$y = \frac{1}{2}\left\{ |x| - |x-n| - \sum_{i=1}^{n} \operatorname{sgn}(x-i) \right\}$ by putting the slope equal to 1, summing

the first part and putting a_i^+ equal to zero and a_i^- equal to 1.

As an example of a continuous function, let us take the curve given in fig. f:

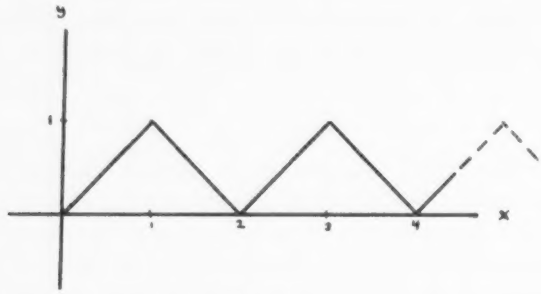


fig. f

where the slope alternates as +1 and -1, the x coordinates are units apart and n is even. Simplifying the basic equation (6), our equation in this case is given by:

$$y = \frac{1}{2}\left\{ |x| - |x-n| \right\} + \sum_{i=1}^{n-1} (-1)^i |x-i|$$

A third example is given in fig. g:

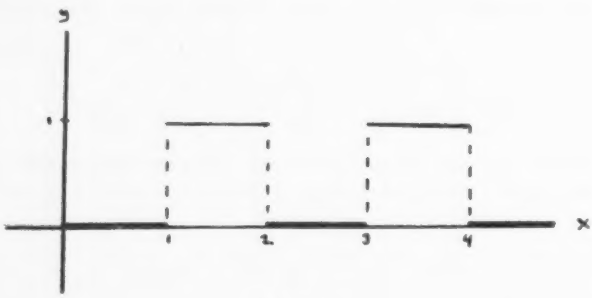


fig. g

the equation of which is given by: $y = \frac{1}{2} \sum_{i=1}^{n} \left\{ \operatorname{sgn}(x-2i+1) - \operatorname{sgn}(x-2i) \right\}$

It should be noted that so far we have restricted ourselves to a finite number of line segments. When the number of points becomes infinite we run into obstacles trying to sum the absolute values, which is not possible except in some very simple cases.

Alfred University

CURRENT PAPERS AN BOOKS

Edited by
H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

The purpose and policies of the first division of this department (Comments on Papers) derive directly from the major objective of the MATHEMATICS MAGAZINE which is to encourage research and the production of superior expository articles by providing the means for prompt publication.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited. Comments which express conclusions at variance with those of the paper under review should be submitted in duplicate. One copy will be sent to the author of the original article for rebuttal.

Communications intended for this department should be addressed to

H. V. Craig, Department of Applied Mathematics,
University of Texas, Austin 12, Texas.

Comment on "A Speedy Solution of the Cubic"

by Robert E. Greenwood

Shortly after the appearance of John T. Pettit's article "A Speedy solution of the Cubic" in the November-December 1947 issue of *Mathematics Magazine*, vol. 21, pp. 94-98, this writer had occasion to compute the zeros of

(1) $f(x) = 60x^3 - 120x^2 + 75x - 14$ and of

(2) $g(x) = 810x^3 - 1620x^2 + 1035x - 208$.

In order to test the usefullness of this new method, the writer decided to use it in computing the zeros of $f(x)$ and $g(x)$. Elementary computations gave $K_f = (-675/64) \simeq -10.55$, $Z_f = 1.14$ and the zeros of $f(x)$ as

$$x_{1f} \simeq 0.768, \qquad x_{2f} \simeq 0.341, \qquad x_{3f} \simeq 0.891 .$$

The writer then used these x values as first approximations to get more accurate values by another method and it was found that

$$x_{1f} \simeq 0.768067, \quad x_{2f} \simeq 0.340972, \quad x_{3f} \simeq 0.890961 .$$

It would appear, therefore, that Pettit's method is capable of giving three significant figure accuracy in some examples.

With respect to $g(x)$, the following results were obtained: $K_g = -28.125$, $Z_g = 1.04$ and

$$x_{1g} \simeq 0.713, \qquad x_{2g} \simeq 0.412, \qquad x_{3g} \simeq 0.876 .$$

Since $\sum_i x_{ig} = 2$ is a necessary consequence of equation (2), some question as to the three significant figure accuracy may well be raised here. Clearly four or more decimal values would be needed to decide this point.

Pettit notes that three real roots exist for the case $K < -6.75$, and both of the examples considered by this author fall within that classification. It would appear to this author that for the ranges $1.00 < Z < 1.50$, $-\infty < K < -6.75$ additional places might well be calculated and thus extend the utility of the tables. (Interpolation near $Z = 1.04$ seems quite inaccurate, four figure values for Z would help.) This suggestion might lead to more than three figure accuracy for the important case of three real roots, and it should not be considered as a criticism of this excellent utilitarian solution of the cubic.

University of Texas

Author's Reply

by John T. Pettit

I have computed tables for the cubic that give roots to five place accuracy, however, these tables are rather long.

Mr. G. H. Comfort (Wilmette, Ill.) has informed me since publication of "A Speedy Solution of the Cubic" that Mr. H. A. Nogrady has a very similar method. (See "A New Method for the Solution of Cubic Equation." Lithograph Print, Edward Bros., Ann Arbor, Michigan).

The solution of the quartic follows directly from the solution of the resolvent cubic.

I append a similar solution for the quintic that depends on the use of a nomograph rather than tables.

Consider the Quintic

1) $y^5 + py^2 + qy + r = 0.$

Write this in the form $y^5 + p(y + \frac{q}{2p})^2 + (r - \frac{q^2}{4p^2}) = 0.$

Let $y = \frac{q}{2p} z.$

Substituting we get $z^5 + \frac{8p^4}{q^3}(z + 1)^2 + \frac{32p^2}{q^5}(r - \frac{q^2}{4p^2}) = 0.$

If we put $K_1 = \frac{8p^4}{q^3}$ and $K_2 = \frac{32p^2}{q^5}(r - \frac{q^2}{4p^2})$ we have

2) $z^5 + K_1(z + 1)^2 + K_2 = 0.$

If we now let $u = z^5$ and $v = (z + 1)^2$ we get

3) $u + K_1 v + K_2 = 0.$

This is seen to be the equation of a straight line in the u, v plane. Eliminating z between u and v we get

4) $v = (u^{1/5} + 1)^2.$

The figure on the opposite page is a graph of equation 4). Observe that 4) is an invariant curve with respect to the coefficients p, q, and r in equation 1, and that it can be assumed linear over small intervals in u for $u < -2$.

A Simple Rule for Solving The Quintic

Compute K_1 and K_2. Lay a straight edge representing the straight line 3) on the figure, and read off u at the points of intersection with curve 4).

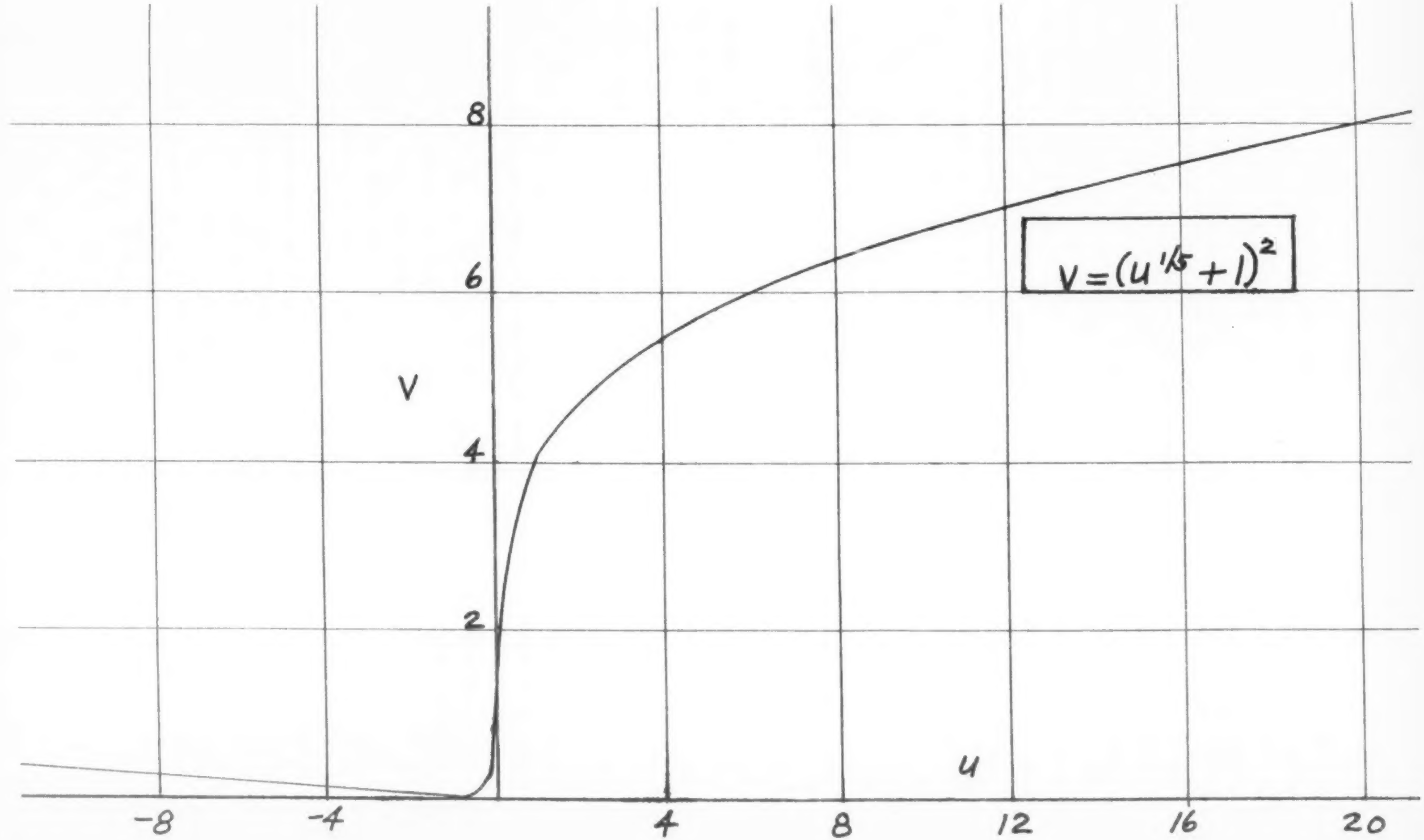
Therefore

$$z = u^{1/5} \quad \text{and} \quad y = \frac{q}{2p} u^{1/5}.$$

(I.B.M. equipment could be used to find accurate values of u and v at points of intersection.)

Example: $x^5 + x^2 - 2x + 3 = 0.$ $K_1 = \frac{8(1)^4}{(-2)^3} = -1,$ $K_2 = \frac{32}{-32}(3 - \frac{4}{4}) = -2,$ and $x = \frac{q}{2p} z = -z.$ Whence $u - v - 2 = 0.$ From the curve $v = 6.40 = (z + 1)^2$, $z = \pm 2.53 - 1 = 1.53, -3.53.$ $u = 8.40 = z^5$, $z = 1.53$ (thus providing a check). Therefore $z = 1.53$ and $x = .1.53$ which is correct to two decimal places.

U.C.L.A.



V = (u^1/5 + 1)^2
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Essential Business Mathematics. By Lewellyn R. Snyder. McGraw-Hill book Co., New York, 1947. XII + 434 pages. Price $2.75.

This book is an effort to provide practical mathematical training for students in business departments of junior colleges who are seeking to obtain employment in the business world, or who wish to provide a background for further work in accounting and other related subjects. The text is divided into two parts; the first part dealing with the essentials of business arithmetic and the second part with the essentials of business mathematics.

Included in the first part are such topics as the mechanics of computations and the fundamentals of problem solving. The second part deals with interest and bank discount, trade and cash discount, retailing, ownership and corporate securities, social securities and wage payments, personal, business and automobile insurance, property taxes, home ownership, averages and graphs. The appendix gives some remedial work in arithmetic with short cuts for multiplication.

The rules and definitions in the book are stated with clarity and conciseness. A large number of illustrative examples are given and each section is followed by a well selected group of problems for solution. The text as a whole is rather elementary and limited in scope. There is no work offered in annuities, amortization of debts, bond valuation or life insurance such as is usually given in texts on financial mathematics. The text does not include the usual tables for compound interest and annuities.

The reviewer detected a few unimportant misprints and errors. The book, though elementary and limited in scope, should prove a worth-while addition to the texts in this field.

A. W. Richeson

Mathematical Aids for the Engineer. By Raymond W. Dull. McGraw-Hill Book Co., New York, 1946. X88 + 346 pages. Price $4.40.

The title of this book indicates the type of person to whom it is addressed and its scope is evident from the table of contents which consists of 28 chapters and 151 sections. The general topics are as follows: uniform scales, logarithms and organic growth, the right - angled triangle, circles, vectors, imaginary and complex numbers, hyperbolic function, several chapters on the various aspects of motion, kinematics, and five chapters on ordinary differential equations.

The major part of the material should be familiar to the average graduate of an engineering school. However, many engineers may find it convenient to have a text, such as this, where they may find the application of the simple theories of mathematics to engineering problems. This book is not a textbook for classroom use, but rather a reference book for the engineer showing the application of the simple topics of algebra, geometry, trigonometry, the calculus and differential equation applied to the practical problems of the engineer.

The subject matter is conveniently arranged for ready reference and the definitions and illustrations are clear and concise. A typical example is the discussion of the relation between the circular and the hyperbolic functions. The explanations are lucid and to the point and are followed by practical examples drawn from the various fields of engineering. In general the many diagrams are well drawn but the chief criticism of the reviewer is that some of the figures are much too small. In the opinion of the reviewer this book should be a valuable addition to the library of the practicing engineer.

A. W. Richeson

Analytic Geometry. By David S. Nathan and Olaf Helmer. Prentice-Hall, Inc., New York, 1947. X + 402 pages.

According to the preface this book seeks to prepare for calculus, engineering, and the physical and social sciences, and also to "develop the student's powers of intuition and rigorous thinking". There are six pages of formulas and tables for reference, nineteen pages of answers to odd-numbered problems, and an index of nearly eight pages. One finds applications to physics, chemistry, astronomy, engineering, and economics. The authors name the formulas appearing throughout the text the "geometry-algebra dictionary". They state that "the book is organized around two themes: equations of loci and loci of equations". Each chapter ends with an excellent summary and set of review problems. The book contains both plane and solid analytic geometry.

The introduction to the process of "proving analytically" geometric theorems seems insufficient. Students find this material difficult. On page 155 in the first line of the footnote for "defiintion" read "definition". Arrowheads on x and y axes are dropped intentionally after the first few pages, but in the chapter on transformations confusion exists as to their use on x and y axes and on x' and y' axes. (See Figures 104, 105, 107, 112, 113.) In Figure 105 the prime for x' is omitted. This reviewer is not keen about the introduction of inversion into the book. However, any teacher is free to omit the subject. The definition of an algebraic curve is correct as far as it goes; but why not give a more nearly complete definition here?

The book discusses carefully the sign used in the reduction of $Ax + By + C = 0$ to the normal form, treats very well the matter of extraneous roots of an equation, and deals thoroughly with absolute value. Degenerate conics are handled carefully. The authors are very good on families of curves and on curve tracing, including the addition of ordinates and a nice example on damped vibration.

Marion E. Stark.
Wellesley College.

The Theory of Mathematical Machines. By Francis J. Murray. King's Crown Press, Morningside Heights, New York, 1947. vii + 116 pages. $3.00.

This book on mathematical machines will be welcome to two classes of readers, those who design and build them and those who desire to use them. Just because this book does not go into all the technical detail required by the designer and builder, it is all the more readable for the potential user. It presents a well organized display of the principles and methods which are available and currently declassified. The footnotes and the bibliography constitute a valuable set of leads to the scattered literature. Since the field is an active one, the reader should be aware of the fact that the current literature and developments are reported in the quarterly journal, "Mathematical Tables and Other Aids to Computation", published by the National Research Council.

The present increasing interest and activity in the field of mathematical machines is based on the fact that they increase the class of computations it is feasible to make for purposes of research, rational design, military studies or other technical problem solving. For these purposes a widely held understanding of basic principles is desirable and this book is well adapted to

the task of dissemination. The excellent forward and lucid style make study easy. The only criticism is that the lack of an analytic index to the contents and the lack of titles under the numerous illustrations make reference tedious.

The subject matter of the book is divided into four parts of four or five chapters each. Part I on "Digital Machines" discusses counters, digital adders, digital multipliers, and punch card machines. Part II on "Continuous Operators" discusses adders, multipliers, integrators and differentiators, amplifiers, and means of representing functions of a single variable. Part III on the "Solution of Problems" discusses examples of similitude solvers, direct calculators, and adjustors. Part IV on "Mathematical Instruments" discusses planimeters, integrometers, integraphs, and harmonic analysers.

To evaluate Prof. Murray's treatment of his subject it is necessary to realize that human beings think and work at a side range of levels of abstraction. The objects of their endeavors, e.g., mathematical machines, can likewise be considered from various levels of abstraction. If one climbs too high one tends to see too much in too little detail, but one can fail to climb high enough. The reader who gives this book the necessary time and effort will find that the wealth of illustrations and detailed discussion and the general statements mutually illuminate each other. He will find that he is thinking at a level of abstraction from which both the theory and technical detail are within easy reach.

N. Grier Parke
Industrial Physicist
Concord, Mass.

Analytic Geometry and Calculus. By John F. Randolph and Mark Kac. The MacMillan Company, New York, 1946. ix + 642 pages.

The preface tells us that this book is printed in two sizes of type and the problems are separated into three groups. The portions of the text in fine print together with the problems of Group III are intended for real students. In the main, Group II problems simply give more practice when added to Group I, but some of them demand a bit of originality. The authors make a selection from the Group III problems for undergraduate mathematics club talks.

The book is very well printed and easy to read. The figures are excellent, especially *Fig*. 70. 1*B* (p. 185). The table of contents is given in considerable detail. We find two pages of trigonometric functions and radians-to-degrees where they are useful for graphs involving trigonometric functions. At the end of the book is a table of integrals, followed by a list of answers to some of the problems and an index.

There is a very careful preparatory study of functions and hints as to maxima and minima before limits and derivatives appear. Limits are treated intuitively. The discussion and problems on relative maxima and minima are excellent. So is the whole treatment of curve tracing. We find an occasional interesting historical note.

The authors are fully aware of those parts of the subject that usually trouble students, and spend time and ingenuity in explaining such points thoroughly. This should save class time for instructors. The book looks ahead into mathematics. Students cannot come to the end of this text with the feeling that they know all there is to know. They have met hints of "ϵ,δ", $y = [x]$, lattice points, the Σ notation, references to articles in the *American Mathematical Monthly*, one suggestion of what Topology does, a brief comment on contour maps, *etc*.

Throughout this book we meet pertinent bits of advice: "tables of integrals are not foolproof"; "this is a very useful trick"; "the student is strongly advised not to use this formula, but in each spedific case to proceed as in example 2".

Twice at least the fact that the authors possess an excellent sense of humor becomes obvious, to the great joy of the reviewer, who is not going to give page references. You will laugh more if you come upon them suddenly for yourselves.

We find a wealth of applications to physics and quite a number to economics, all taken up with clarity and thoroughness.

The authors distinguish very frankly between a rigorous proof and an intuitive study of a topic. We think this will be attractive and helpful to both instructors and students.

There seem to be very few errors. On p. 88 in line 14 an l has been omitted. On p. 93 in problem 8 read "problem 7" for "problem 5". On p. 105, fifth line below *Fig.* 43. 1, read "$v_4 - u_4$" for "$u_4 - v_4$". On p. 353 in the middle of the page read "$\left(\frac{\Delta y_k}{\Delta x_k}\right)^2$" for "$\left(\frac{\Delta y_k}{\Delta x_k}\right)$". On p. 446 in line 2 read "R_2 " for "R_2 ". *Etc.*

There are always some suggestions a reviewer feels inclined to offer to authors. In connection with $y = \frac{\sin x}{x}$ why not mention what the curves $y = 1/x$ and $y = -1/x$ do to the graph? It seems to make the horizontal asymptote more easily understood. Why not show with a figure the reason why the parabola, ellipse, and hyperbola are called "conics"? Why not name the graphs of *Fig.* 109.6 to 109.10? Why not introduce the name "one-to-one correspondence" the first time the subject comes up (*i.e.* on p. 11 with the linear coordinate system) and then use the name later with both plane rectangular coordinate systems and rectangular coordinate systems in space, instead of having the name appear only in the third and last instance? Why not give the names of the epicycloid and hypocycloid (p. 402) and of the three polar coordinate curves of *Fig.* 153.1, *Fig.* 153.2, and *Fig.* 157.4? Why not introduce the term "indeterminate forms"? Oh well, why carp? The book is excellent.

Marion E. Stark.
Wellesley College.

Solid Analytic Geometry. By J. M. H. Olmstead.
New York, Appleton-Century Co., 1947. 13 + 257 pages. $4.00.

The nine chapters of this book cover the conventional material (lines, planes, second degree surfaces, algebraic curves) without the use of vectors or homogeneous coordinates. Right-handed rectangular axes are used throughout, except for one article on cylindrical and spherical coordinates.

The novel feature is the use of matrices: rank, elementary transformations, characteristic roots and vectors of a matrix, orthogonal matrices, transpose of a matrix, inverse of a non-singular matrix, sums and products of matrices. The theory of matrices is applied to such topics as: the classification of systems of planes, families of points, the invariants of a quadratic surface and the reduction by translations and rotations of the general second degree equation to canonical form.

The interested student should find plenty of food for thought in this text even if his instructor, pressed for time, is forced to omit many of the starred

articles. It would be desirable that more books be published in which vector and matrix theory, as well as other mathematical theories used in modern applications, are introduced at the lowest possible course level.

W. E. Byrne

Plane Trigonometry. By Elmer B. Mode.
Prentice-Hall, Inc., New York, 1947. x + 200 pages + tables, 16 pages. $2.40.

The author, a professor in Boston University, states that this text is an enlargement and revision of a lithoprinted work which has gone through four editions. Chapters I - VII deal with the subject of plane trigonometry and three supplementary chapters discuss approximate computation, logarithms, and the slide rule. This arrangement facilitates the introduction of the supplementary material at the will of an instructor. The author himself uses the chapters on approximate computation and logarithms at the beginning of his course in trigonometry.

Chapter I is largely devoted to an explanation of angular measure by the degree, the radian, and the mil.

Chapter II defines the six usual trigonometric functions of any angle in standard position in terms of the abscissa, ordinate, and radius vector of a point on its terminal side, and calls attention to the changes in sign dependent on the quadrant in which its terminal side lies. Special definitions of functions of an acute angle in a right triangle are given, and functions of angles of 30°, 60°, 45°, 0°, and 90° are specifically treated. The use of tables with respect to functions of acute angles is explained.

In Chapter IV the formulas for reducing functions of angles greater than 90° are neatly obtained.

Chapter V studies the graphs of trigonometric functions and emphasizes their usefulness in physical applications.

In Chapter VI the standard trigonometric identities and formulas are derived, the addition formulas for the sine and cosine in an unusual and interesting way by use of the distance formula of analytic geometry. Ample practice in proving identities and in solving equations of condition is given here.

The supplementary chapter on approximate computation presents clearly and concisely material which is often inadequately treated in texts. The chapter on logarithms introduces the use of numbers in standard form. The third supplementary chapter gives a brief but careful treatment of the slide rule.

The tables included in the book are four-place tables. A section in the chapter on logarithms is, however, devoted to an explanation of the use of logarithmic tables of greater accuracy.

Answers are given to odd-numbered exercises only.

This book is well arranged, formulas and illustrations are numbered for convenient reference by chapter, miscellaneous topics can be omitted without affecting the continuity of the subject, a sufficient number of exercises of wide variety are included. The text is recommended as a significant addition to trigonometry texts.

Helen G. Russell.
Wellesley College.

Edited by

Joseph Seidlin, L. J. Adams and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, ...ition, curriculum, tests and measurements, and any other topic related to teaching, ... invited. Papers on any subject in which you, *as a teacher*, are interested, or ...tions which you would like others to discuss, should be sent to Joseph Seidlin, ...red University, Alfred, New York.

THE MEANING OF PLANE GEOMETRY

As part of our primitive heritage, we tend to react adversely to what is ...ifferent." In our thinking, it is an easy transition from "different" to "strange" ... "foreign" to "enemy," and as soon as anything is classified as "enemy," we ...ect it and attempt to destroy it. When things take on an air of familiarity, ...no longer fear them—what we know and understand, we accept. Apparently, then, ...ual understanding is an indispensable factor for living in peace. It is ...fortunate that the means to achieve understanding should be the double-edged ...ol of language which not only gives man the power to communicate ideas, but is, ... itself, so liable to misuse and misinterpretation that its effectiveness is ...iously limited. Scientists have, to a certain extent, learned to overcome the ...itations of language. The precision of science demands a mode of expression ...ich is concise and admits of only a single possible interpretation. The ...uation $E = mc^2$ means the same thing to scientists everywhere. It has the ...ther advantage of eliminating emotional associations. The drawback, of course, ... that it is intelligible only to the initiated.

Moreover, science has developed an approach to acquiring knowledge and solving ...oblems which is, so far, the most adequate man has been able to devise. ...iefly, the method consists of the following steps:

1. A statement of the problem with all terms clearly defined and limited.
2. Collection of all facts related to the problem by research and experiment.
3. A sifting of the facts to eliminate any which may be irrelevant.
4. Careful observation of the data collected to determine what significant conclusions may justly be drawn.
5. Acceptance of conclusions based on such procedures whether or not they are in agreement with previously held theories.

...his impartiality is a most important factor in the scientific method.)

Plane geometry is one attempt, usually the first encountered by the student, to ...ain the mind to adapt itself to this type of thinking. It is an introduction to ...gical reasoning on an elementary level and uses the properties of familiar ...gures as its subject matter. There is a distinct advantage in the use of ...iangles, squares, circles, and such, inasmuch as we can develop our method ...ith them without complication by emotional bias which might interject itself ...to the problem were we to use less objective material. We can, at the same ...me, learn many useful properties of these figures.

Geometry probably antedates recorded history. We know that the Babylonians ...ed triangles and parallel lines, and their carriage drawings indicate division ... the circle into four and six equal parts. To the Egyptians, as far back as ...00 B.C., geometry was a practical science of measurement which enabled them to ...ake out boundaries of fields obliterated by seasonal inundation of the Nile.

The rope-stretchers who measured square corners by means of twelve knots evenly spaced on a rope (see diagram) were the forerunners of Oythagoras who established the theorem which bears his name and generalizes the relationship between the sides of a right triangle. The Greeks took this system of measuring plane figures, refined it and generalized it, finally establishing the rigid system outlined in Euclid's "Elements of Plane Geometry." The word geometry itself derives from the Greek, gaia, ge = the earth, and metrein = to measure.

$$3^2+4^2=5^2$$
$$9+16=25$$

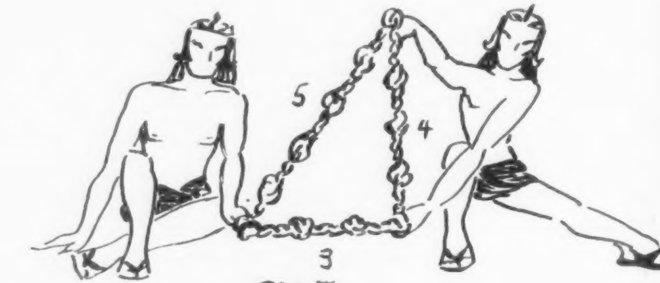


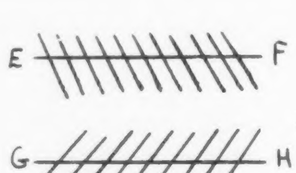
FIG. I

In building a method of systematized thinking, we need a plan. We must agree about certain fundamentals so that our finished product will be useful. It must be broad enough to be applicable to other fields, yet limited enough to make a coherent system of thinking.

Where shall we start? Ideally, we should like to begin at the very beginning of things, accepting nothing dogmatically so that everything we discover could be looked upon as Absolute Verity. However, since it is unlikely that any of us can know what constitutes Ultimate Truth, if there be such a thing, we shall have to leave that matter to the philosophers, and, starting somewhere in our common experience, make compromises and agreements, and be content to verify each step as we proceed on the basis of these agreements. This verification we call proof.

What is proof and why is it necessary? Proof is necessary because the human senses are not infallible. Many things are not what they seem to be and we are all familiar with optical illusions—railroad tracks that appear to run together in the distance—and objects that apparently take on properties from their surroundings which they do not inherently possess.

Lines AB and CD are the same length.



Lines EF and GH are parallel

Fig. 2

A proof, in some cases, may be an appeal to authority. The evidence presented by an expert witness in a court of law constitutes such a proof. It is, however, reliable only in proportion to the expertness of the authority consulted. Proof may be given by testimonial. "Ask the man who owns one," is an appeal to the experience of a person who, presumably, has had extended acquaintance with the qualities of the product so advertised. Sometimes we accept conclusions drawn from situations similar to the ones we are considering. For example, an argument for disarming the police in New York might be supported by the statement, "The police of London are unarmed. If it works in London, it should work in New York." This is reasoning by analogy and is valid, provided a sufficiently large number of the contributing circumstances are alike in the two situations compared. All of these methods of proof are useful, on condition that we recognize and allow for the limitations under which they operate validly.

However, a geometric proof is more rigorous than these and is a process of reasoning, based, not on observation, nor on authority which is sometimes untrustworthy, but based on argument from *stated specific assumptions, through a series of tested logical steps, to an inevitable conclusion.* Such a conclusion is no stronger than the assumptions on which it is based.

In setting ourselves the task of establishing a system of thinking which makes use of the properties of space, our first problem is to answer the question, "What is the nature of space, or how shall we think of it?" Is it endless or finite? If it is infinite, can a straight line extend infinitely far in both directions or does it curve so that the two ends really meet? (In this case, the curvature would be so small that the portion of the line we are able to observe would still look straight.) Every conclusion we develop will depend on the fundamental concepts we adopt concerning the nature of space. Actually, we are forced to admit that we do not know the space we have to deal with in the sense that we feel that we know the wetness of rain or the softness of fur. We have to accept certain properties of this space because our experience justifies such acceptance as reasonable and workable. So far as plane geometry is concerned, we assume that there is such a thing as a flat surface, as represented by a drawing board, which has length and width, and that straight lines may be drawn on this plane surface and extended as far as we wish in either direction. We agree that since, by assumption, two-dimensional surfaces exist, simpler elements exist similarly: the line, which has one dimension, length, and the point which has no dimensions at all, but merely indicates position. It is also convenient to assume that certain processes may be carried out in this space; that figures may be moved without changing their size or shape and that a line may be rotated about any fixed point on it. Beyond these elements, point, line, surface, and the processes of motion and rotation, we shall be careful to define every additional term we use.

In order to be sure that our terms are clearly understood, we must first establish standards for our definitions. A good definition must tell, not only what an object is, but also what it isn't. To say that a pencil is a writing tool is a good start, but it doesn't tell that it isn't a pen, or a crayon, or a piece of chalk. Yet, it is better than saying that a pencil is a piece of wood with a core of lead in it. We should also like our definitions to read correctly backwards as well as forwards, so that an object which satisfies the conditions of the definition may be designated by the word defined, that is, if a triangle is a three-sided figure, then every three-sided figure must be a triangle. We can summarize the

requirements of a good definition as folloss:

1. It must place the object defined in the smallest group to which it belongs (the writing tool rather than the piece of wood).
2. It must furnish sufficient detail to distinguish that object from all similar objects in that group but should give the minimum number of details that will do so.
3. It must be reversible.
4. The terms used in the definition should already have been defined or be so simple as to require no further definition in themselves.

These criteria for a good definition are applicable in fields other than geometry and, if used, would settle many arguments. The effort to define precisely clarifies thinking and often leads to the discovery that the participants in an argument are not discussing the same subject.

All definitions are agreements, and, therefore, acceptable as facts in proofs. Besides definitions which are added from time to time, there are other types of general statements which are part of our common experience and so fundamental as to be acceptable without proof. If a pair of identical twins, each weighing 120 pounds, occupy positions on a seesaw at the same distance from the center, the seesaw will be in balance. Should one of the twins kick off his shoes, the balance is destroyed; as soon as the other twin kicks off his shoes, the balance is restored. The same state of affairs would exist if each twin were to pick up one of a pair of identical basketballs, or if a second pair of twins were to climb on the seesaw, doubling the weight on either side. If two objects are equal and we add or subtract equal quantities, or multiply (double, treble, halve) both by the same number, the results in every case are also equal—the balance is maintained. Such obvious relationships we call *axioms*—statements concerning quantities in general which are accepted as true without proof. Geometry makes use of several axioms. In any good cook book there is usually found a page which lists substitutions which may safely be made. Three tablespoons of cocoa and one tablespoon of fat are the equivalent of one ounce of baking chocolate. Evidently, then, in any recipe calling for baking chocolate, we can substitute the cocoa and fat in the given amounts and achieve the same results. "*A quantity may be substituted for its equal.*" Using the same reference, we find that, instead of one cup of fresh sweet milk, it is possible to use ½ cup of evaporated milk plus ½ cup of water, or ¼ cup dried skim milk powder plus one cup of water and three teaspoons melted fat, or one cup skim milk plus two teaspoons melted fat. These are all equivalent to each other and may be used interchangeably without affecting the quality of the product. "*Quantities which are equal to the same or to equal quantities are equal to each other.*" Anyone who has ever worked a jigsaw puzzle is familiar with the axiom, "*The whole of a quantity is equal to the sum of its parts and is greater than any one of them.*" Some of the axioms are so freely accepted in everyday situations that many geometry texts do not even list them. For example, anything is equal to itself (the identity) and one quantity is either greater than, equal to or less than another; in other words, these are all the possibilities, so far as we know. There is also a complete set of axioms for inequalities. These are illustrated by the following examples, where $>$ means ' is greater than ' and $<$ means ' is less than ':

$$\begin{array}{l} 5 < 7 \\ \underline{2 = 2} \text{ (adding)} \\ 7 < 9 \end{array} \qquad \begin{array}{l} 8 > 4 \\ \underline{2 = 2} \text{ (subtracting)} \\ 6 > 2 \end{array} \qquad \begin{array}{l} 5 < 7 \\ \underline{2 = 2} \text{ (multiplying)} \\ 10 < 14 \end{array} \qquad \begin{array}{l} 8 > 4 \\ \underline{2 = 2} \text{ (dividing)} \\ 4 > 2 \end{array}$$

$$\begin{array}{l} 3 < 5 \\ \underline{2 < 4} \text{ (adding)} \\ 5 < 9 \end{array} \qquad \begin{array}{l} 6 > 4 \\ \underline{3 > 2} \text{ (adding)} \\ 9 > 6 \end{array} \qquad \begin{array}{l} 5 = 5 \\ \underline{3 > 2} \text{ (subtracting} \\ 2 < 3 \end{array}$$

These examples show that *if equals are added to or subtracted from unequals or if unequals are multiplied or divided by the same positive number, the inequality is maintained,* i.e., the results are unequal in the same order as the original quantities. It is also evident that *if unequals of the same order are added together, the results have the same order of inequality, while if unequals are subtracted from equals, the order of inequality is reversed.*

Further, consider a set of nested drinking cups, 4 oz., 6 oz., and 8 oz., respectively. Since the 8 oz. cup contains the 6 oz., and the 6 oz. contains the 4 oz. cup, then the 8 oz. cup contains the 4 oz. cup. In algebraic symbols, if $A > B$ and $B > C$, then $A > C$.

In addition to the axioms which are self-evident relations between quantities in general, we state certain fundamental assumptions which are accepted without proof. They might be called our working hypotheses. An example that comes to mind of such a set of basic agreements is contained in the Declaration of Independence. "We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain inalienable Rights, that among these are Life, Liberty and the Pursuit of Happiness." It was not stated that the signers had verified by experience or experiment that these assumptions are essential to a harmonious society, but they held them to be "self-evident."

In geometry, there are general relations concerning geometric quantities which are assumed to be true without proof. The eyepiece of a bombsight is equipped with cross-hairs so that the target may be spotted accurately. The intersection of two lines determines one and only one point. A rifle, on the other hand, frequently has two sights, so that the target may be "lined up." This demonstrates the geometric principle that two points determine one and only one straight line. On a checkerboard, it is possible to go from one corner to another diagonally opposite in any number of moves. The most economical, however, is a series of moves along the squares which lie in a straight line between the two corners, *since a straight line segment is the shortest segment joining two points.*

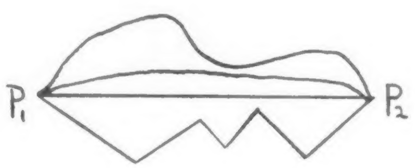


Fig. 3

The moving checker itself is an example of another assumption, namely, that a *geometric figure may be moved without changing its size or shape.* The wires which

form the spokes on the wheels of a bicycle are all the same length, not only in each wheel, but in both wheels. We accept as a fact that *all radii of the same or of equal circles are equal.* Such geometric assumptions as these are called *postulates.*

Somewhere in the course of our pursuit of geometric principles, we shall be confronted with the problem of drawing a circle. A cow tethered to a stake will trace a circle if she walks along a path that is always the maximum distance the rope will allow. Evidently, then, to draw a circle, we need a fixed distance (the rope) and a place to anchor it (the stake), and we shall assume that a circle may always be drawn with a given point as center and a fixed line segment as radius.

Incidentally, this brings up the problem of tools. In plane geometry, we are permitted, in addition to writing materials, only two aids in constructing figures: an unmarked straightedge to draw lines and a pair of compasses with which to draw circles. It is true that we could accomplish more with a more elaborate set of implements, but by tradition and agreement, plane geometry is confined to those constructions which may be made with these two devices. Part of the colorful history of the subject is associated with the limitations involved in the use of these tools. The three classic problems which occupied the thoughts of many of the early Greek geometers, leading them to incidental discoveries in the field, were: (1) The trisection of any angle. (2) The quadrature of the circle (constructing a square equal in area to a given circle). (3) The duplication of the cube (constructing a cube having twice the volume of a given cube). It was finally proved in the nineteenth century that these three tasks are impossible of accomplishment with the tools of plane geometry. They may, however, be solved by the use of other instruments.

With the statement of our *axioms* and *postulates* and the acceptance of our undefined elements—*point, line, plane*—we are now ready to develop our system of reasoning about plane figures. The simplest figure, other than the straight line, which can be formed on a plane surface occurs when two straight lines intersect. Such a figure forms four *angles* (the symbol $\angle$ is used to designate 'angle'). Evidently, there is some relationship between these angles and we shall investigate it. First, following our pattern of procedure, we must define some terms. *An angle is a geometric figure formed by the intersection of two lines*; the point of intersection is called the *vertex* of the angle and the lines form its *sides.*

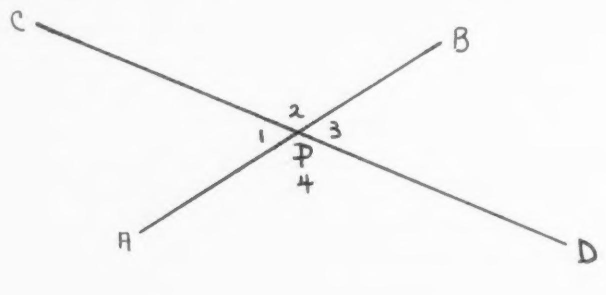


Fig. 4

There are two possible relative positions which pairs of angles assume in this picture. They are either next to each other or they are opposite each other.

In the former case, we call them *adjacent* angles (as $\angle 1$ & $\angle 2$ or $\angle 2$ & $\angle 3$) and in the latter, *vertical* angles (as $\angle 1$ & $\angle 3$ or $\angle 2$ & $\angle 4$). It appears that the pairs of vertical angles might be equal (i.e., $\angle 1 = \angle 3$, $\angle 2 = \angle 4$). We shall use this as an example to show what we mean by a geometric demonstration. Before we do this, however, it will be necessary to define a straight angle, and to outline the steps required in a formal proof. A *straight angle* is an angle whose sides extend in opposite directions from the vertex and lie on a straight line. In the figure, *AB* is a straight angle with vertex at *P*. As can be seen, a straight angle is a straight line with a vertex indicated somewhere on it.

A formal demonstration is the proof of some geometric principle or property and follows a specific, prescribed form. The following elements are essential:

1. A statement of the proposition to be proved.
2. A lettered diagram showing the figures involved.
3. A breakdown of the proposition into hypotheses (what is given) and conclusion (what is to be proved) in terms of the figure drawn.
4. The body of the proof, consisting of a list of statements, each accompanied by a reason which is an axiom, a postulate, a definition, or the result of a previous proof.
5. A final statement which must agree with the conclusion as stated in the original proposition.

Let us apply this to the proof of the simple geometric principle noted above.

Proposition: Vertical angles are equal.

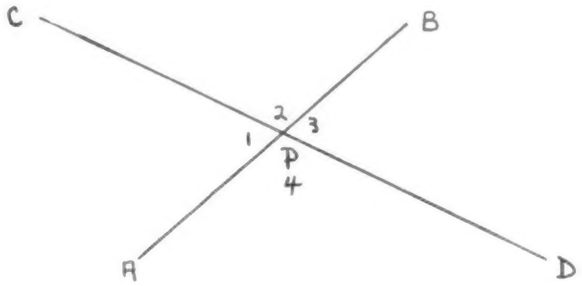


Fig. 5

Given: Straight lines *AB* and *CD* intersecting at *P*.
To Prove: $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$.

Statements	*Reasons*
1. AB and CD are straight lines intersecting at *P*.	1. Given
2. *AB* and *CD* are straight angles.	2. Definition of straight angle.
3. $\angle 1 + \angle 2$ straight angle $\angle 3 + \angle 2$ straight angle	3. The whole of a quantity equals the sum of its parts.
4. $\angle 1$ = str. angle minus 2 $\angle 3$ = str. angle minus 2 (subtracting $\angle 2$ from both sides, above)	4. If equals are subtracted from equals the remainders are equal.

5. $\angle 1 = \angle 3$	5. Quantities equal to the same quantity are equal to each other.

Exactly the same proof shows that $\angle 2$ and $\angle 4$ are equal.

A proposition once proved is called a *theorem* (although sometimes this word is used interchangeably with proposition).

When two lines intersect so that the four angles formed are all equal to each other, we say that the lines are *perpendicular* to each other (the symbol $\perp$ is used to indicate perpendicularity) and the angles are called *right angles*. An angle smaller than a right angle is said to be *acute* (sharp) while an *obtuse* (blunt) angle is one which is greater than one right angle but less than two right angles. For convenience in measuring, we divide the right angle into 90 equal angles, calling each part one *degree*. An acute angle would, therefore, have fewer than 90 degrees (90°) and an obtuse angle would lie between 90° and 180°. A straight angle, according to this system of measurement, contains 180°.

The type of proof given above, which advances directly from hypothesis to conclusion, and appears to have sprung full-blown from the head of Jove, is called a synthetic proof. The word "synthesis" implies a building up, putting the elements together to form something more complex. Obviously, complete proofs don't occur to us as soon as a proposition is stated. The demonstration of a geometric principle in its final form is a synthetic proof, but the process which indicated where to start and what steps to take was quite the opposite; it was a tearing down process, an analysis. Briefly, an analytic proof starts with the conclusion and examines the conditions which, if true, will verify that conclusion. Suppose the conclusion is C. We examine it and discover that C is true if B is true, but B is true if A is true and we know A to be true because it depends on an axiom. Then we "synthesize" a proof, beginning with A, advancing through B to C.

Sometimes, even after analyzing a problem, it is difficult to arrive at a suitable beginning for a direct proof, that is, one which advances from hypothesis to conclusion in successive steps. In this case, we frequently arrive at our conclusion indirectly by listing all possible conclusions in the given case and eliminating all but one by showing that the others lead to contradictions of established principles or of fundamental assumptions. This is the type of reasoning a student uses when confronted with a multiple-choice question on an examination. For example: 'The woman who ruled England during Shakespeare's lifetime was (1) Marie Antoinette (2) Victoria (3) Elizabeth.' There are three possibilities. Assuming that we cannot select the correct one immediately, let us consider them in turn. Suppose the answer to be Marie Antoinette. This is impossible, since Marie was a French queen and didn't rule England. Next, consider Victoria. Victoria was still ruling at the beginning of the twentieth century and we know Shakespeare died in 1616. Victoria could not have ruled almost 300 years. The only remaining possibility is Elizabeth, and since there is one correct answer and we have eliminated the other two, this must be the proper response to the question. We did not establish directly that Elizabeth was Queen of England in Shakespeare's lifetime, but, rather, indirectly, by showing that the other possibilities contradict known facts or lead to absurd conclusions. The indirect method of proof is, in fact, sometimes called "reductio ad absurdum" (reduction to absurdity). The essential feature of the method is to assume something temporarily as true, then, show that such an assumption leads to an impossible conclusion, for if a correct line of reasoning leads to an incorrect

ult, the assumption on which it is based must be false.

To establish all the properties of plane figures and the relationships between m is beyond the scope of this brief survey of the subject. There are, however, rain basic concepts which give direction to much of the subject development geometry and it seems fitting that we should examine some of these. A major t of geometry is concerned with closed straight-line figures. The angle, which considered as the simplest geometric figure, is an open figure and its sides can extended indefinitely without changing its essential property, namely, that it the measure of the difference in direction of two lines. The minimum number lines required to make a closed figure is three, and the figure so formed is lled a *triangle*. The triangle is the simplest closed figure belonging to the ass of many-sided figures known as *polygons*. We exhibit below a few specific lygons and their familiar names.

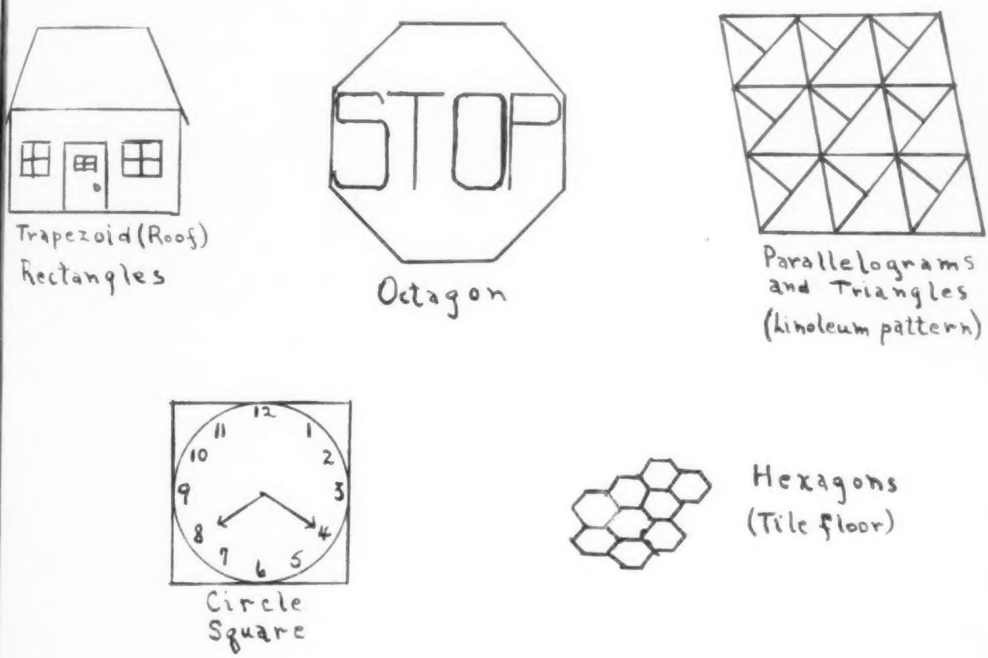
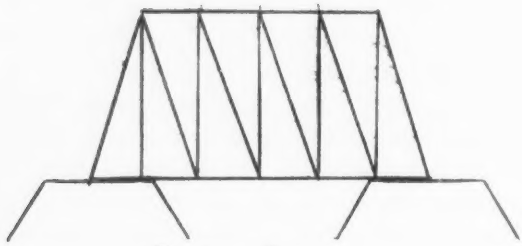


Fig. 6

The triangle possesses certain properties which make it useful in construcion. Anyone who has braced a shelf by means of a bar of wood from the edge of he shelf to the wall, made use of the important fact that a triangle is a rigid gure and cannot be changed in size or shape by pressure exerted on the sides or ertices. This is not true of other polygons, and where they are used in onstruction, cross-pieces are inserted to convert them into triangles. This roperty of rigidity stems from the fact, that, given three pieces of wood, he triangle formed from them is unique and any other triangle formed from hree like pieces of wood will have exactly the same size and shape as the



Bridge Span

Fig. 7

first one. In mathematical language, this relationship of having the same size and shape is called *congruence*—the property which enables two or more figures to coincide, when one is superimposed on the other. A stack of dinner plates is an example of congruence, each one fitting exactly with the ones beneath and above it. The symbol $\cong$ is used to denote congruence and implies equality of size and similarity of shape. A little experimentation will suggext that only one triangle can be constructed when we have given: *three sides; two sides and the angle included between them;* or *two angles and any side,* and it is easily proved that this is true. Instead of the twelve knots of the rope stretchers, we use this property of congruence as the basis for the construction of right angles. With compasses set at P, we mark off equal lengths on either side of P, so

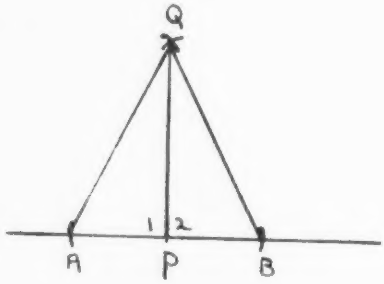


Fig. 8

that PA PB. Taking a slightly larger radius, we strike equal arcs from A and from B, intersecting at Q. Triangles PAQ and PBQ have three sides equal respectively, and are, therefore congruent. The corresponding angles, $\angle 1$ and $\angle 2$, are equal, and since their sum is a straight angle, each one is a right angle.

Since congruence imposes conditions of similarity and equality, the question arises whether either similarity of shape or equality of area can occur independently.

Similarity of shape obviously occurs in photographs and maps. The photograph which reproduces you possesses the same features but very much reduced in size. Every line must take the same direction with respect to adjoining lines to preserve the general contours, and the reduction in size must be uniform, that is, if your arm in the picture is 1/10 the actual length of your arm, then every other feature must appear as 1/10 its actual length. We can state these observations in geometrical terms by saying that two figures are similar if the corresponding angles are equal and the corresponding lines are proportional. Similarity is made use of in the reverse order from that used in making photographs when building a house from a set of plans. A considerable part of geometry is devoted to establishing minimum conditions for the similarity of triangles and other polygons. It can be shown that two triangles are similar *if their corresponding sides are proportional,* or *if two sides are proportional and the included angles equal,* or merely *if two angles of one are equal to two angles of the other.* This last set of conditions is related to the fact that the sum of the angles of any triangle is a straight angle (or 180°), as can be demonstrated by cutting a triangle from paper and then tearing off the angles and putting them together in any order. It is apparent that any *two right triangles are similar if one acute angle of one is equal to the corresponding acute angle of the other,* since each triangle already has a right angle. This fact is the basis for much of the work of elementary trigonometry.

We consider now equality of areas without necessarily having similarity of figures. Two figures which cover the same amount of surface are equal. They may, however, differ in shape. A table scarf may measure 4" by 16" or 8" by 8"; in either case, it will cover 64 sq. in. of table surface, but in one case it is rectangular and in the other, square. In order to determine whether or not two figures are equal in area, there must be some method of measuring area and one section of geometry deals with developing formulas for finding the areas of plane figures such as the triangle, rectangle, parallelogram, etc. Most polygons can be divided up into simpler figures such as triangles or rectangles, so that formulas for these suffice to cover the subject of area. Waiting for an elevator, one sometimes unconsciously counts the tiles in the floor. This is the method for working out the basic problem in area. If the floor is rectangular in shape, it is soon discovered that it is not necessary to count the tiles individually, but merely to count the number in each row and multiply by the number of rows. Hence we arrive at a formula for the area of a rectangle: base × height, or in symbols, $A = bh$. Since two triangles are produced by drawing a diagonal of the rectangle, each is half the area and so, as a formula for the area of a triangle, we have: $A = \frac{1}{2}bh$. The area of a parallelogram is found in the same manner as that of a rectangle. A trapezoid is composed of two triangles which have different bases, and the formula for its area is $A = \frac{1}{2}h\,(b + b')$. Any regular polygon is composed of as many equal triangles as it has sides hence its area is given by the formula $A = \frac{1}{2}hp$, where h is the radius of the inscribed circle and p is the perimeter (See Fig. 10). It is interesting to consider the problem of finding the area of an irregular plane figure. A close approximation to the area can be made by assuming the small sections to be rectangles. Then the area of each rectangle is b times

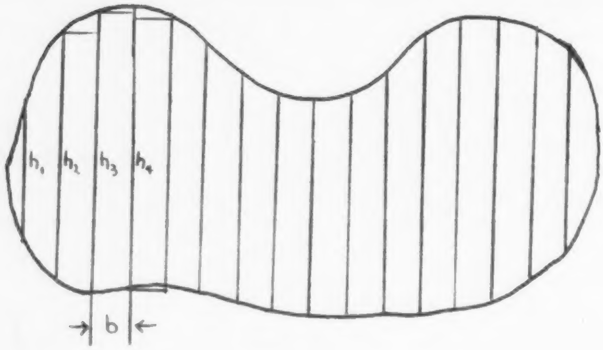


Fig. 9

its height, and the total of all these areas is $b(h_1 + h_2 + h_3 + \cdots + h_n)$. The smaller b is taken, the closer the sum of the areas of the rectangles approaches the actual area. This is essentially the method of the integral calculus which considers this sum as b approaches zero. In geometry, we use this device to determine the area of a circle. It can be shown that when the circumference of any circle is divided by its diameter, the result is always the same. The Greeks designated this constant by the letter π, and it has an approximate value of 3.1416. Since this is so, the formula for the circumference of a circle is $C = \pi d$ in terms of the diameter or, $C = 2\pi r$, in terms of the radius. If we continue to increase the number of sides of a regular inscribed polygon, (See Fig. 10) it will approach the circle in size and shape. The area of the circle

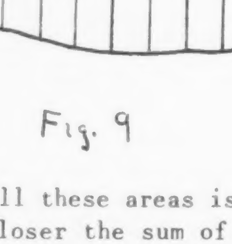


FIG. 10

will then be approached by the area of a regular polygon, and the radius of the circle will be approached by the perpendicular from the center of the circle to any side of the polygon. Now the area of a regular polygon is $\frac{1}{2} \times h \times p$ where h is the perpendicular and p is the perimeter (i.e. the sum of the sides) of the polygon, and the perimeter (circumference) of a circle is $2\pi r$. Hence the area of a circle is $\frac{1}{2} \times r \times 2\pi r$ or $A = \pi r^2$.

With an understanding of postulational thinking, and the concepts of congruence, similarity and equality, such as we have sought to convey, readers who desire to know more of the machinery of plane geometry will find satisfaction in reading any text on the subject.

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MATHEMATICS AS A THERAPY

by Irving M. Cowle

In the early stages of World War II news came of a new type of program to aid hospitalized service personnel to a speedier recovery by providing worthwhile and constructive activities during the convalescent period.

The Army Air Forces, through its Convalescent Training Program, established a number of classes that patients in hospitals might attend. There were offered a wide range of subjects from English, history and mathematics to a mild amount of physical education for the ambulatory patients. The bedridden were given a tutoring service in any subject in which they might evidence interest, including the choice of many craft and art projects.

The men who participated in this novel curriculum were, for the most part, patients suffering from some physical disability. In the middle of 1944 the Army Air Forces were confronted with a problem of a different nature. Men were pouring back from overseas suffering from "combat fatigue", the World War II term for war-shattered nerves and loss of confidence. These soldiers were, generally speaking, not mentally ill; but unless given the proper guidance and treatment immediately could easily lapse into cases of a more serious nature.

To handle this problem, the Air Forces established a group of Convalescent Hospitals, scattered throughout the country, where these borderline returnees would be rested and assisted in reestablishing their self-confidence and mental balance. The personnel who were to work in these establishments were carefully chosen and, prior to assignment, given a program of training and orientation to acquaint them with the peculiar situation they were about to enter, a picture of the type of curriculum to be followed, and an understanding of the problems of the soldier patients.

The best of medical and psychiatric care, as well as excellent mess and recreation facilities was available at these hospitals. In addition, the usual range of courses in photography, music, art, crafts and other vocational subjects was offered. As an experiment, the men, if interested, were to choose courses in the academic line—mathematics, English and history being among those available. Though this was something of a departure from the standard curriculum in such a situation it was desirable to offer the patients courses in whatever fields they might show interest.

The fundamental procedure in reference to mathematics, was to give the patients opportunity to discover that they could handle capably problem solving and follow processes of reasoning similar to those encountered in the day to day contacts outside the hospital. This was far more challenging, and reassuring than the time-honored block of wood and carving knife, the so called "paper doll" type of therapy which has for so long been prevalent. The result was that the patient, who has been recaining his confidence through logical problem solving and thinking, is the better prepared to meet the obstacles he must face outside. On his day of discharge, for example, he will not be overwhelmed by the task of deciphering a time table to locate the proper train to take home.

The problems posed by this unprecedented situation will undoubtedly be of great interest to teachers of mathematics. The author was fortunate to be appointed the mathematics instructor at the first of these Convalescent Hospitals, established at Ft. Thomas, Ky.

There were many complications which made this program far different from any teaching situation previously encountered. Any patient who so desired could take one or more courses in mathematics; these could be taken during any period

of the day. Every subject from arithmetic to calculus in which any interest was indicated was offered. The men could begin at any time and at any stage of any course they wished; there was no compulsory attendance nor did the student have to work for a complete period but could leave the classroom at any time. Under these conditions it can readily be seen that the scheduling of classes for definite periods was an impossibility.

With these limitations, the following method of handling the classroom situation seemed the most feasible; and practice bore out that assumption Tables were furnished at which the patients could choose any seat—no attempt being made to limit those studying a certain subject to a particular section of the room. United States Armed Forces Institute textbooks, originally designed for self teaching, were procured for every subject in all grades from high school to college. Paper and pencils were placed on every table.

When a new student entered the class, which happened every day in every subject, he was shown the various texts and allowed to browse and choose the particular subject he felt he would like to study, assistance being given if requested. Once this choice had been made, a paper bearing the student's name was inserted in the pages as a bookmark.

With a little assistance in starting, the student soon became accustomed to working by himself; summoning the instructor at any time he desired help. For the particular situation this system was far superior to the continuous compulsory concentration which group instruction demands.

Whenever a student felt inclined to leave, he merely placed a bookmark, with his name protruding, in the page on which he was working. The text was then left on the table. Each evening, all books were alphabetized with reference to the student's names so that on the following day, the students could easily locate their books and commence working at the very spot reached on the previous day. The men derived definite satisfaction out of being independent of the instructor in performing these functions. Strangely enough, once the routine was established, students rarely left before the end of the period and many took advantage of their privilege of working longer.

Let us examine for a moment some of the results of this system, keeping in mind the war-harrasses, discipline-weary, and nervous type of individual for whom it was designed. Because they were able to come and go at will, to study anything desired for as long as they were interested and only when they felt so inclined, the majority of the patients carried full schedules. Many patients completed courses and were given final examinations which, if passed, were accepted by most high schools for credit towards a diploma. The soldiers who accomplished this goal felt the time at Ft. Thomas was particularly well spent.

More valuable and more difficult to measure, is the confidence which many regained in their mental capacities. The psychiatrists had naturally been reassuring, but nothing is so conclusive as actually doing problems and finding that one can complete thinking processes as well as he had previously been able to do—and then push on to master more complex processes.

For the "borderline" patient, who if not given the proper help and treatment might degenerate into a serious mental problem, this use of mathematics is quite practical. In fact, to such a patient the "paper doll" treatment indicates to him that he is considered to be in faulty mental condition, whereas the purpose of this program is to assure him that his mind is perfectly healthy and capable.

From the favorable results obtained by the Army Air Forces with this revolutionary type of treatment, one begins to wonder if there are not many patients in our state institutions who would benefit by realistic challenge.

MATHEMATICAL MISCELLANY

Edited by

Marian E. Stark

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematically" will be welcome. Address: MARIAN E. STARK, Wellesley College, Wellesley 81, Mass.

Professor Cleon C. Richtmeyer reports that Central Michigan College of Education (Mount Pleasant, Michigan) had a total enrollment of 761 in mathematics for the fall semester of 1947-1948. "This is not necessarily 761 different students, since one student might be taking two or more classes." Can any other teachers' college or college of education report a larger number?

Try the following Trigonometric Scrambles at the next meeting of your Mathematics Club. They were sent in by Professor Elmer B. Mode of Boston University.

1. Stance	6. Cunnoift
2. Stainmas	7. Nadira
3. Tietynid	8. So nice
4. Mathgolir	9. Quart dan
5. Itraceschartic	10. Alengrit

Excerpts from Letters

a) I am no Professor Emeritus, but your inquiry on "what to retire to" was most interesting to me. I would like to suggest to those interested in mathematics, that the exploration and development of duodecimals has proved to be a most absorbing hobby to me, and it might prove equally interesting to them.

Ralph H. Beard.

b) The format of the Mathematics Magazine is very pleasing, and the papers include very pleasant reading. I am particularly glad to see that college students are to be given a chance to publish research papers as well as "professionals".

I am enclosing a check for $7.00. Since I have already paid for a year at $3.00, and since I was not aware of sponsoring subscriptions, I hope you will include me. I will also bring the magazine to the attention of the mathematicians here with whom I am associated.

c) This conversation took place (in all seriousness) on April first.

Is it true that zero times infinity equals one?
Well, what do you mean by "infinity"?
The largest possible number.
Is there a largest possible number?
Yes, but they have not found out what it is, yet.

The "they" and the "yet" seem very rich to me!

William R. Ransom.

d) Isn't class enthusiasm the best remedy for the lethargy of the C student about whom you asked in the Jan.-Feb. issue of this magazine?

Calculated interest in the subject will not, alone, do the trick. The spirit required is this interest hallowed by the teacher's personal interest in his students and his unfaltering faith in their possibilities.

Pressure devices may make good examination-passers and seperate those who have the tenacity to survive from the more timid but it takes consuming enthusiasm to lift the c student out of the caste to which he feels doomed, when one meets him only three hours per week and he is under the pressure of a schedule heavy for him.

Glenn James.

In the January 1948 number of *Mathematical Tables and other Aids to Computation* will be found several very interesting items. See in particular the first of two articles by Franz L. Alt (pages 1-13), entitled "A Bell Telephone Laboratories' Computing Machine—I". Also "A New Approximation to π" (pages 18, 19) by D. F. Ferguson of the University of Manchester in England and John W. Wrench, Jr. of Washington, D. C., and an enlightening quotation from the *Soviet News*, Soviet Embassy in London on "Mathematics as a theoretical weapon"(pages 65 and 66).

The Mathematics Institute at Duke University. By Helen S. Collins, High School, Brookline, Mass.

For the past seven summers teachers of high, junior-high and college mathematics from over half the states in the union have attended the Mathematics Institute at Duke University in Durham, North Carolina. During the ten-day period these teachers studied intensively problems of common interest. They learned new uses of mathematics, discussed methods of enriching the subject and obtained a broader concept of its place in our educational program.

The work of the Institute centers about the Mathematics Laboratory, the purpose of which is to make available in one place a wide range of materials relating mathematics to science, industry, engineering and commerce. Included in this are samples of students' work such as models, drawings and instruments contributed by teachers attending the Institute.

The general theme of last summer's Institute was "Mathematics at Work." Many of the noon and evening lectures given by representatives of various branches of industry revealed that the mathematics used in their particular fields ranged from simple arithmetic to the most complicated differential equations. Visits to nearby industrial plants gave members an insight into some of the mechanisms used.

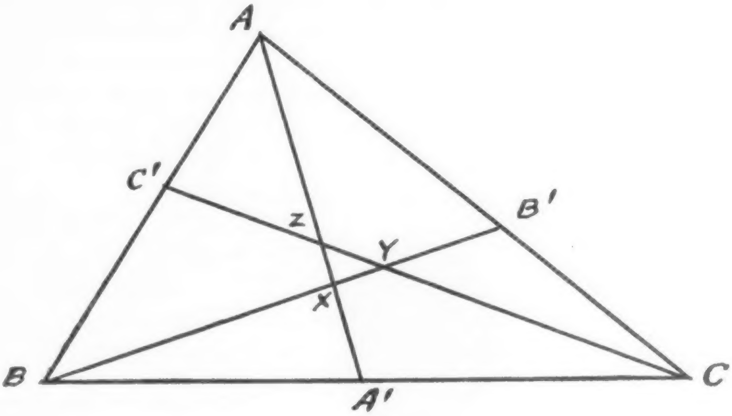
The main work of the Institute is carried on through study groups, among the most popular of which this past summer were those discussing "Aids in the Study of Geometry" and "The Enrichment of mathematics."

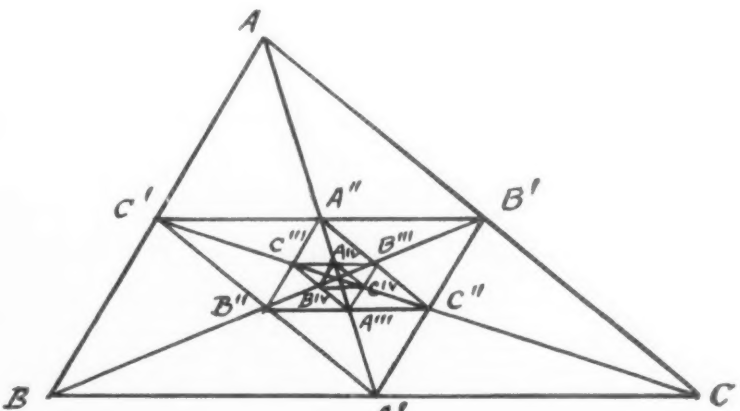
Social life is amply provided for and throughout the Institute there is a friendly, genial atmosphere. It is apparent that members are there not only to obtain inspiration but to share with others their own methods and experience.

For more detailed accounts of the study groups and lectures, consult "The Mathematics Teacher" for February 1947 and March 1948 .

Proof of the Theorem on the Intersection of the Medians of a Triangle. By E. F. Canady.

So far as the writer is aware, the following is a new proof of the familiar theorem that the medians of a triangle meet in a point that is two-thirds the distance from any vertex to the midpoint of the opposite side. Let us suppose that the medians do not meet in a single point. Then they will meet in pairs in three points X, Y, Z, forming a triangle *finite* in size. Join the midpoints of the given triangle to form triangle $A'\ B'\ C'$; in a similar manner we form triangles $A''\ B''\ C''$, $A'''\ B'''\ C'''$, etc.





Now the medians of triangle ABC are also the medians of triangles $A'\ B'\ C'$, $A''\ B''\ C''$, etc.; hence their intersections X, Y, Z must lie in each triangle $A^n\ B^n\ C^n$. But each triangle of the series has sides which are half as long as the sides of the preceding triangle; hence a triangle may be found which is smaller than triangle $X\ Y\ Z$, which was supposed to be contained in each triangle $A^n\ B^n\ C^n$. Therefore the supposition that the medians do not meet in a common point was false. Let us call their intersection P.

Now $AP = AA'' + A''\ A^{IV} + A^{IV}\ A^{VI} + \cdots$. Set $AA' = m$; then $AA'' = \frac{m_a}{2}$; $A''\ A^{IV} = \frac{m_a}{8}$; $A^{IV}\ A^{VI} = \frac{m_a}{32}$; etc.

Hence $AP = \frac{m_a}{2} + \frac{m_a}{8} + \frac{m_a}{32} + \cdots$, or $AP = \frac{2}{3}m_a$.

Meredith College

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Current Papers and Books
(Comments and Reviews)

MATHEMATICS MAGAZINE

Problems and Questions

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Our Contributors

(*B*iographic Sketches)

*Proposals numbered 600 or over were published in the National Mathematics Magazine, the predecessor of the Mathematics Magazine.